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On the Lattice-based Sum and Its Consequences for the Construction of Associative Aggregation Operators

A Thesis Submitted in partial fulfillment of the requirement of the degree of Master of Science in Engineering Mathematics

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ABSTRACT

Associative aggregation operators on bounded lattices are special aggregation operators that have proven to be useful in many fields like fuzzy logics, expert systems, neural networks, data mining, and fuzzy system modeling. Nullnorms, uninorms, t-norms, t-conorms, and many other operations all belong to the class of associative aggregation operators. One of the typical constructions for associative aggregation operators on the unit interval [0,1] is the ordinal sum construction. As observed, in general, an ordinal sum construction may fail on a general bounded lattice. Motivated by the last observation, a new sum-type construction called lattice-based sum has been recently introduced by El-Zekey *et al*. [30]. In this thesis, based on the lattice-based sum of (bounded) lattices indexed by a (finite) lattice-ordered index set, new methods for constructing nullnorms and uninorms on bounded lattices, which are lattice-based sums of their summand sublattices, are developed. Subsequently, the obtained results are applied for building several new nullnorm and uninorm operations on bounded lattices. As a by-product, the lattice-based sum constructions of t-norms and t-conorms obtained by El-Zekey [31] are obtained in a more general setting where the lattice-ordered index set need not be finite and so-called t-subnorms (t-subconorms) can be used (with a little restriction) instead of t-norms (t-conorms) as summands. Furthermore, new idempotent nullnorms on bounded lattices, different from the ones given in [16], have been also obtained. We point out that, unlike [16], in our construction of the idempotent nullnorms, the underlying lattices need not be distributive.

PUBLICATION PAPERS

This thesis is based on the work described in the following papers:

1. M. El-Zekey and M. Khattab, Lattice-based Sum Construction of Nullnorms on Bounded Lattices, Circulation in Computer Science, 3 (2018) 1-9.

2. M. El-Zekey and M. Khattab, Lattice-based sum construction of uninorms on bounded lattices, Submitted (2018).

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INTRODUCTION

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1.1 General

Associative aggregation operators on the unit interval are special aggregation operators that have proven to be useful in many fields like fuzzy logics, expert systems, neural networks, data mining, and fuzzy system modeling. t-norms, t-conorms, uninorms, nullnorms and many other operations all belong to the general class of associative aggregation operators (see e.g., [10]).

Associative aggregation operators have been also studied on some more general structures, for example, bounded partially ordered sets and bounded lattices, stimulating some investigations in topology and logic. One of typical constructions for associative aggregation operators on the unit interval is the ordinal sum construction. There were several attempts to generalize this construction method considering a general bounded lattice. As observed (see e.g., [67]), in general, an ordinal sum construction may fail on a general bounded lattice.

Inspired by the last observation, a new sum-type construction, called lattice-based sum, has been recently introduced [30]. It is a generalization of the ordinal sum technique. This is done by allowing for lattice-ordered index set instead of linearly ordered index set. The aim of the present research is to propose new methods, based on the lattice-based sum, to construct various associative aggregation operators on bounded lattices such as t-norms, t-conorms, uninorms and nullnorms.

1.2 Problem statement

In recent years, several methods for constructing new associative binary operations on the unit interval from given associative binary operations were proposed, all resembling, yet differing from the ordinal sum of t-norms. There have been several attempts to generalize the ordinal sum construction considering a general bounded lattice (see e.g., [34, 56, 57, 65, 67]), inspired first by Goguen's proposal to consider fuzzy sets with membership values from bounded lattices. However, these methods have long been blamed for their limitations in constructing new associative aggregation and their inability to cope with a general bounded lattice. On one hand, as observed in [67], ordinal sum construction may not work on bounded lattices. On the other hand in [67], there exist ordinal sum t-norms on bounded lattices which are not an ordinal sum of some of their sublattices. Summarizing, there is a need for a new sum-type construction generalizing the ordinal sum construction and coping very well with associative aggregation operators on general bounded lattices. One possibility is the lattice-based sum based on lattice-ordered index set [30]. Note that, in [30], the focus has been on lattice-based sums of either posets or lattices as summand structures only. In this thesis, we will investigate and develop some new methods, based on the lattice-based sum approach, for constructing various associative aggregation operators on bounded lattices.

1.3 Objectives

The long-term goal of the research is to develop general methods, based on the lattice-based sum scheme to construct various associative aggregation operators on bounded lattices such as t-norms, t-conorms, uninorms and nullnorms. The result of this study would open new aspects for the investigation of aggregation functions on bounded lattices. It would also be useful in obtaining associative operations suitable for human thinking/evaluation, in several applications.

1.4 Thesis outlines

This thesis is organized in six chapters as follow:

Chapter One: Presents a brief introduction on the subject of the thesis, the objectives, and the motivations.

Chapter Two: Shows an overview of the lattice-based sum technique for building new posets and lattices from given ones.

Chapter Three: Presents a literature survey on the most important associative aggregation operators, their definitions, properties, and different construction methods on bounded lattices.

Chapter Four: Contains our proposal for the construction of nullnorms as well as idempotent nullnorms, t-norms and t-conorms on bounded lattices.

Chapter Five: Contains our proposal for the construction of uninorms as well as idempotent uninorms, t-norms and t-conorms on bounded lattices.

Chapter Six: Summarizes the major results of this study and provides recommendation for future work.

The published papers from this thesis are [33] for nullnorms and [32] for uninorms.

CHAPTER TWO

LATTICE-BASED SUM OF BOUNDED LATTICES

Chapter two
Lattice-based sum of bounded lattices

2.1 Introduction and preliminaries

In the literature, there were several methods on how to build new ordered structures from simpler ones such as the *disjoint union* of ordered structures [19], the ***ordinal sum of posets in the sense of Birkhoff*** [5, 67] (it is also referred to as ***linear sum of posets*** [19]). The ***horizontal sum of bounded posets*** [5, 19] and the ***lattice-based sum of posets and lattices*** [30].

As observed in [30], the lattice-based sum technique generalized the well-known ordinal sum of posets in the sense of birkhoff by allowing for ***lattice-ordered index set*** instead of ***linearly-ordered index set***. It is pointed out in [30] that the lattice-based sum technique extends also the horizontal sum of bounded posets based on unstructured index set.

In this chapter, we review the lattice-based sum technique for building new posets and lattices from simpler ones. We start by some concepts concerning posets and lattices.

Definition 2.1: ([5], [19])

Let L be a set, an order (or partial order) on L is a binary relation $\leq$ on L such that for all $x, y, z \in L$,

i. $x \leq x$ (Reflexive)

ii. $x \leq y$ and $y \leq x$ imply $x = y$ (Antisymmetric)

iii. $x \leq y$ and $y \leq z$ imply $x \leq z$ (Transitive)

A set L equipped with an order relation $\leq$ is said to be a ***partially ordered set*** (*Poset* for short).

Definition 2.2: ([5], [19])

Let L be an ordered set and let $S \subseteq L$. An element $x \in L$ is an *upper bound* of S if $s \leq x$ for all $s \in S$.

Definition 2.3: ([5], [19])

Let L be an ordered set and let $S \subseteq L$. An element $x \in L$ is a *lower bound* of S if $s \geq x$ for all $s \in S$.

Definition 2.4: ([5], [19])

Let L be an ordered set and let $S \subseteq L$. The set of all *upper bounds* of S is denoted by S^u and the set of all *lower bounds* of S is denoted by S^l, defined as follow

$$S^u = \{x \in L | (\forall s \in S)\ s \leq x\} \quad \text{and} \quad S^l = \{x \in L | (\forall s \in S)\ s \geq x\}$$

Definition 2.5: ([5], [19])

If S^u has a least element x, then x is called the *least upper bound* of S, dually, if S^l has a greatest element x, then x is called the *greatest lower bound* of S. These two elements are obviously unique for each S.

The least upper bound of S is sometimes called *supermum* of S and is denoted by $Sup\ S$. The greatest lower bound of S is also called *infimum* of S and is denoted by $inf\ S$. We write $x \vee y$ (reads "x join y") in place of $sup\{x, y\}$; when it exists and $x \wedge y$ (reads "x meet y") in place of $inf\{x, y\}$; when it exists.

Definition 2.6: ([19])

Let L be an ordered set. Then L is a *chain* if, for all $x, y \in L$, either $x \leq y$ or $y \leq x$ (that is, if any two elements of L are comparable). Alternative names for a chain are *linearly ordered set* and *totally ordered set*.

Definition 2.7: ([5], [19])

Let L be a non-empty ordered set. If $x \vee y$ and $x \wedge y$ exist for all $x, y \in L$, then L is called a *lattice*.

Definition 2.8: ([5], [19])

Let L be a lattice and $\emptyset \neq M \subseteq L$. Then M is a *sublattice* of L if for all $a, b \in M$ implies $a \wedge b \in M$ and $a \vee b \in M$.

Definition 2.9: ([5], [19])

A *bounded lattice* is a lattice $(L, \leq, \perp, \top)$ which has the top and bottom elements written as: $\top$ and $\perp$, respectively, that is, there exist $\perp, \top \in L$ such that $\perp \leq x \leq \top$, for all $x \in L$.

Definition 2.10: ([5], [19])

Let $(L, \leq, \perp, \top)$ be a bounded lattice and let $a, b \in L$. If a and b are *incomparable* (i.e., $a \nleq b$ and $b \nleq a$), we write $a \parallel b$.

Definition 2.11: ([5], [19])

Let $(L, \leq, \perp, \top)$ be a bounded lattice and let $a, b \in L$, where $a \leq b$. A subinterval $[a, b]$ of L, is a sublattice of L defined as

$$[a, b] = \{x \in L | a \leq x \leq b\}$$

Similarly, $]a, b] = \{x \in L | a < x \leq b\}$, $[a, b[= \{x \in L | a \leq x < b\}$, $]a, b[= \{x \in L | a < x < b\}$.

Example 2.1:

All ordered structures in Figure 2-1 are examples of bounded lattices.

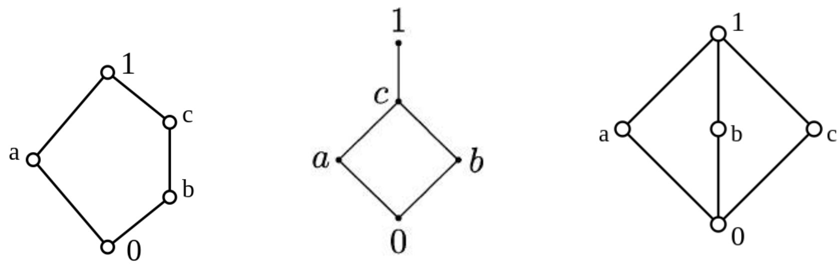

Figure 2-1 Bounded lattices Examples

Definition 2.12: ([5])

A lattice $(L, \leq)$ is a *distributive lattice* if it satisfies one (or, equivalently, both) of the distributive identities

i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

for all $x, y, z \in L$.

It has been shown in [5] that in a distributive lattice, for all $x \in L$, if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$, then $x = y$.

2.2 Lattice-based sum of bounded posets

In this section, we review the lattice-based sum technique for building bounded posets from the given ones. First, we would like to list all standard customary notations when we deal with lattice-based sums, as follow:

$(\Lambda, \sqsubseteq)$ denotes a finite lattice-ordered index set in which each two element subset $\{\alpha, \beta\}$ has an infimum denoted by $\inf\{\alpha, \beta\}$, and a supermum denoted by $\sup\{\alpha, \beta\}$. For each $\alpha \in \Lambda$, $(L_\alpha, \leq_\alpha, \bot_\alpha, \top_\alpha)$ denotes a bounded partially ordered set (poset) with a top element $\top_\alpha$ and a bottom element $\perp_\alpha$ for some $\alpha \in \Lambda$. Lowercase Latin letters (e.g. "x","y" and "z") are used as variables ranging over the elements of L_α, and lowercase Greek letters (e.g. "α","β" and "γ") are used as variables ranging over the elements of Λ. If $\alpha, \beta \in \Lambda$ such that $\alpha \sqsubseteq \beta$ but $\alpha \neq \beta$, then we will write $\alpha \sqsubset \beta$. The cardinality (the number of elements) of a set L will be denoted by $|L|$.

Remark 2.1:

In [30], a lattice-ordered index set need not be finite and each summand poset need not be bounded. But, in this thesis, and from a practical point of view, we restrict our consideration to finite the lattice-ordered index set, and to bounded summand only.

Definition 2.13: ([30])

Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$. The *Λ-sum family* is a family of bounded posets $\{(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)_{\alpha\in\Lambda}$ that satisfies the following: for all $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ the sets L_α and L_β are either disjoint or satisfy one of the following two conditions:

i) $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$ with $\alpha \sqsubset \beta$, where $x_{\alpha\beta}$ is both the top element of L_α and the bottom element of L_β and where for each $\varepsilon \in \Lambda$ with $\alpha \sqsubset \varepsilon \sqsubset \beta$ we have $L_\varepsilon = \{x_{\alpha\beta}\}$ and for all $\delta, \gamma \in \Lambda$ with $\delta \parallel \gamma$, $\delta \sqsubset \beta$ and $\alpha \sqsubset \gamma$ we have $L_\delta = \{y_{\delta\gamma}\}$ or $L_\gamma = \{z_{\delta\gamma}\}$ where $y_{\delta\gamma}$ is the top element of $L_{\inf\{\delta,\gamma\}}$ and $z_{\delta\gamma}$ is the bottom element of $L_{\sup\{\delta,\gamma\}}$.

ii) $1 \leq |L_\alpha \cap L_\beta| \leq 2$ with $\alpha \parallel \beta$, and for each $x_{\alpha\beta} = L_\alpha \cap L_\beta$, $x_{\alpha\beta}$ is the top element of both L_α and L_β and the bottom element of

$L_{\sup\{\alpha,\beta\}}$, or $x_{\alpha\beta}$ is the bottom element of both L_α and L_β and the top element of $L_{\inf\{\alpha,\beta\}}$.

Note that, the *Λ-sum family* in Definition 2.13 where referred to as the *Λ-sum family of bounded posets* while the *Λ-sum family of bounded lattices* for those whose all underlying bounded posets L_α are bounded lattices was denoted by $\left((L_\alpha, \wedge_\alpha, \vee_\alpha)\right)_{\alpha\in\Lambda}$ where $\wedge_\alpha$ and $\vee_\alpha$ are the meet and join operations on L_α, respectively.

Definition 2.14: ([30])

Let $(\Lambda, \sqsubseteq)$ be a finite lattice-ordered index set and let $\{(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)\}_{\alpha\in\Lambda}$ be a Λ-sum family. The *lattice-based sum* $\bigoplus_{\alpha\in\Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$ is the set $L = \bigcup_{\alpha\in\Lambda} L_\alpha$ equipped with the order relation $\leq$ defined by:

$$x \leq y \text{ if and only if}$$
$$\begin{cases} \exists \alpha \in \Lambda \text{ such that } x, y \in L_\alpha \text{ and } x \leq_\alpha y \\ \text{or} \\ \exists \alpha, \beta \in \Lambda \text{ such that } (x, y) \in L_\alpha \times L_\beta \text{ and } \alpha \sqsubset \beta \end{cases} \tag{2.1}$$

This type of lattice-based sum where referred to as *lattice-based sum of bounded posets*.

Theorem 2.1: ([30])

With all assumptions of Definition 2.14, the lattice-based sum $(L, \leq, \perp, \top) = \bigoplus_{\alpha\in\Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$ is a bounded partially ordered set.

Note that, the strategy just described focuses on the *union* of the carriers and an order consistent with both the order of the underlying posets and the order of the lattice-ordered index set. Thus, the order relation for elements from different summand carriers is inherited from the lattice-ordered index set.

Remark 2.2:

As shown in [30], if the lattice-ordered index set in Definition 2.14 is a *chain*, then the lattice-based sum reduces to the ordinal sum, i.e., we obtain the ordinal sum of posets in the sense of Birkhoff, in which any two posets overlap in at most one point (see [5] and [67]).

Remark 2.3:

The lattice-based sum in Definition 2.14 extends also the horizontal sum of bounded posets as we can see in Proposition 2.1.

Recall that a bounded poset $(X, \leq, \perp, \top)$ is called a *horizontal sum* of the bounded posets $\big((X_i, \leq_i, \perp, \top)\big)_{i\in I}$ if $X = \bigcup_{i\in I} X_i$ with $X_i \cap X_j = \{\perp, \top\}$ whenever $i \neq j$, and $x \leq y$ if and only if there is an $i \in I$ such that $\{x, y\} \subseteq X_i$ and $x \leq_i y$.

Proposition 2.1: ([30])

Let $(L, \leq, \perp, \top)$ be a bounded poset. Then the following are equivalent:

i) $(L, \leq, \perp, \top)$ is a horizontal sum of the bounded posets $\big((L_i, \leq_i, \perp, \top)\big)_{i\in I}$.

ii) $(L, \leq, \perp, \top)$ is a lattice-based sum of the bounded posets $\big((L_\alpha, \leq_\alpha, \perp, \top)\big)_{\alpha\in\Lambda}$, where $(\Lambda, \sqsubseteq)$ is the lattice in which Λ is the set I with two more elements $\perp_\Lambda$ and $\top_\Lambda$ such that $L_{\perp_\Lambda} = \{\perp\}$ and $L_{\top_\Lambda} = \{\top\}$ and the partial order $\sqsubseteq$ is defined on Λ as:
for all $\alpha \in \Lambda, \perp_\Lambda \sqsubseteq \alpha$ and $\alpha \sqsubseteq \top_\Lambda$.

Example 2.2:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ in Figure 2-2. Then each of the families associated with the structures in Figures 2-3 and 2-4 forms a Λ-sum family of bounded posets. Hence each of these structures is a lattice-based sum of bounded posets.

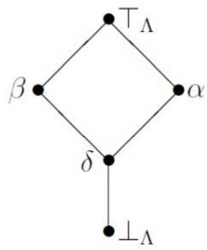

Figure 2-2 The lattice $(\Lambda, \sqsubseteq)$ of Example 2.2

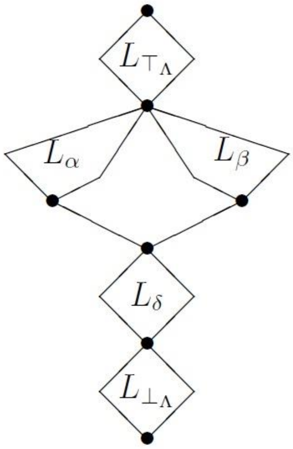

Figure 2-3 The Λ-sum family 1 of Example 2.2

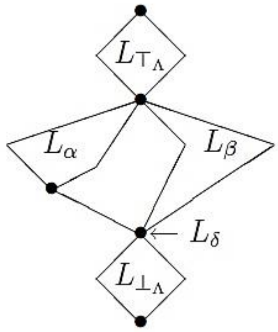

Figure 2-4 The Λ-sum family 2 of Example 2.2

Remark 2.4:

In Figure 2-3, we have $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$, where $x_{\alpha\beta}$ is the top element of both L_α and L_β and the bottom element of $L_{\sup\{\alpha,\beta\}}$ where $\sup\{\alpha, \beta\} = \top_\Lambda$. This satisfies condition (ii) in Definition 2.13. Also, in

Figure 2-4, we have $L_\beta \cap L_{\perp_\Lambda} = \{x_{\beta\perp}\}$, where $x_{\beta\perp}$is the top element of $L_{\perp_\Lambda}$ and the bottom element of L_β and for $\perp_\Lambda \sqsubset \delta \sqsubset \beta$ we have $L_\delta = \{x_{\beta\perp}\}$ which is a *singleton* poset. This satisfies condition (i) in Definition 2.13.

Example 2.3:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ in Figure 2-5. Then the family of bounded posets associated with the structure in Figure 2-6 is not a Λ-sum family because $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$ with $x_{\alpha\beta} = \top_\alpha = \perp_\beta$, $\delta \sqsubset \beta, \alpha \sqsubset \gamma$ but neither $L_\delta = \{\top_{\inf\{\delta,\gamma\}}\}$ nor $L_\gamma = \{\perp_{\sup\{\delta,\gamma\}}\}$. Hence the structure in Figure 2-6 is not a lattice-based sum for the lattice-ordered index set of Figure 2-5 and for the family of bounded posets of Figure 2-6. The main reason is that the order relation is not consistent with the order of the index set, since for $x \in L_\delta$ and $y \in L_\gamma$, we have $x \leq y$ while the only elements δ and γ in the index set associated with x and y, respectively, are incomparable elements in Λ. a slight modification is by putting $L_\delta = \{\top_{\inf\{\delta,\gamma\}}\}$ and hence we get the Λ-sum family of bounded posets associated with the structure in Figure 2-7 in which the order consistency holds, such that, for $x \in L_\delta$ and for $y \in L_\gamma$ we have $x \leq y$, $x \in L_\delta \cap L_{\perp_\Lambda}$ and $y \in L_\gamma$ and hence there exist $\perp_\Lambda, \gamma \in \Lambda$ associated with x and y, respectively, such that $\perp_\Lambda \sqsubset \gamma$.

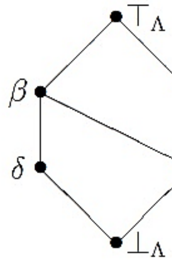

Figure 2-5 The lattice $(\Lambda, \sqsubseteq)$ of Example 2.3

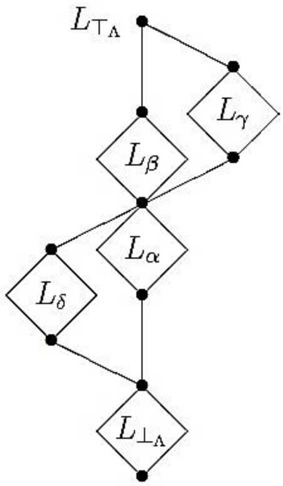

Figure 2-6 Not a Λ-sum family

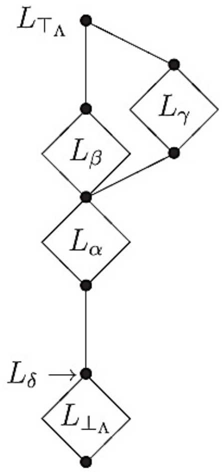

Figure 2-7 Λ-sum family of Example 2.3

For more illustrative examples, we refer to [30].

2.3 Lattice-based sum of bounded lattices

In the previous section, we recalled the main results concerning the lattice-based sum of bounded posets supported by some examples for clarification. In the current section, we will recall the main results concerning the lattice-based sum of bounded lattices.

Definition 2.15: ([30])

Given a lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a Λ-sum family $\{(L_\alpha, \leq_\alpha)\}_{\alpha\in\Lambda}, x \in \bigcup_{\alpha\in\Lambda} L_\alpha$. We say that an element $\alpha^* \in \Lambda$ is a *maximal (minimal) index* of x if α^* is a maximal (minimal) element of the set $I_x = \{\alpha \in \Lambda | x \in L_\alpha\}$. Denote by I_x^{max} and I_x^{min}, respectively, the set of all maximal and minimal indices of x.

Example 2.4:

Obviously, if $\{(L_\alpha, \leq_\alpha)\}_{\alpha\in\Lambda}$ is a Λ-sum family with finite lattice-index set Λ, then, for all $x \in \bigcup_{\alpha\in\Lambda} L_\alpha$, the set $I_x = \{\alpha \in \Lambda | x \in L_\alpha\}$ contains maximal and minimal elements.

Note that, given a Λ-sum family $\{(L_\alpha, \leq_\alpha)\}_{\alpha\in\Lambda}$ and $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \neq y$, we write $x \parallel y$, if for all $\alpha, \beta \in \Lambda$ such that $x \in L_\alpha$ and $y \in L_\beta$ we have $\alpha \parallel \beta$. Also, we write $x \parallel_\alpha y$ if $x, y \in L_\alpha$ for some $\alpha \in \Lambda$ such that $x \nleq_\alpha y$ and $y \nleq_\alpha x$. Obviously, x and y are incomparable if $x \parallel y$ or $x \parallel_\alpha y$ for some $\alpha \in \Lambda$.

Lemma 2.1: ([30])

Let $(\Lambda, \sqsubseteq)$ be a finite lattice-ordered index set and let $\{(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)\}_{\alpha\in\Lambda}$ be Λ-sum family of bounded posets. If $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \parallel y$, then

i) For all $\alpha_1, \alpha_2 \in I_x^{max}$ and $\beta_1, \beta_2 \in I_y^{max}$, $\top_{\inf\{\alpha_1,\beta_1\}} = \top_{\inf\{\alpha_2,\beta_2\}}$.

ii) For all $\alpha_1, \alpha_2 \in I_x^{min}$ and $\beta_1, \beta_2 \in I_y^{min}$, $\perp_{\sup\{\alpha_1,\beta_1\}} = \perp_{\sup\{\alpha_2,\beta_2\}}$.

Example 2.5:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ in Figure 2-8 and the family associated with the structure in Figure 2-9. It is easy to check that the family in Figure 2-9 is a Λ-sum family. Let x be the top of L_{β_2}, y be the top of L_{α_2} and z be the top of L_{α_1}. Then:

i) It is obvious that $x \parallel y$ where $I_x = \{\beta_2, \delta_2\}$ and $I_y = \{\alpha_2, \alpha_3, \delta_3\}$. For $\beta_2, \delta_2 \in I_x$ and $\alpha_2, \delta_3 \in I_y$ (note that β_2 and α_2 are not maximal), we have,

$$\top_{\inf\{\beta_2,\delta_3\}} = \top_{\beta_1} \neq \top_{\alpha_1} = \top_{\inf\{\delta_2,\alpha_2\}}$$

Of course (see Lemma 2.1), if we replace α_2 by the maximal α_3 and replace β_2 by the maximal δ_2, it will render the equality, such that, in this case, we have

$$\top_{\inf\{\delta_2,\delta_3\}} = \top_{\delta_1} = \top_{\inf\{\delta_2,\alpha_3\}}$$

ii) For x and y as described above,

$$\inf\{x, y\} = \top_{\inf\{\delta_2,\delta_3\}} = \top_{\delta_1} = \top_{\inf\{\delta_2,\alpha_3\}},$$

where α_3 and δ_3 are maximal indices of y while δ_2 is the maximal index of x. Although,

$$\top_{\inf\{\beta_2,\delta_3\}} = \top_{\beta_1} = \top_{\inf\{\beta_2,\alpha_3\}},$$

where β_2 is not maximal, we see that

$$\inf\{x, y\} \neq \top_{\inf\{\beta_2,\delta_3\}} = \top_{\inf\{\beta_2,\alpha_3\}}$$

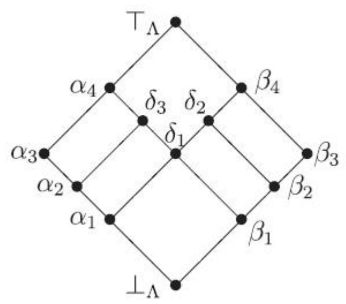

Figure 2-8 The lattice $(\Lambda, \sqsubseteq)$ of Example 2.5

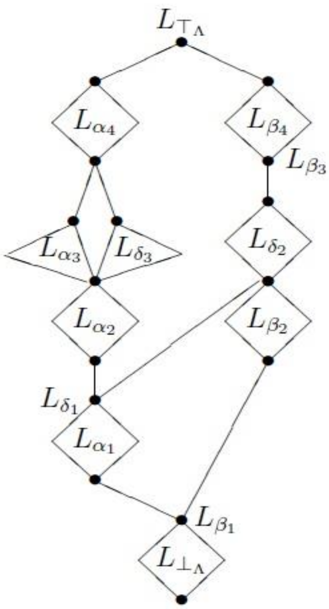

Figure 2-9 The Λ-sum family of Example 2.5

Remark 2.5:

As pointed out from [30], the consecutive repetition of standard ordinal and horizontal sum constructions is covered by the lattice-based sum approach, but the opposite is not true, as we can see in the obtained Λ-sum family in Figure 2-9, although this family is a Λ-sum family, but it is impossible to describe this family as repetition of ordinal and horizontal sums. For more details, we refer to [37, 41, 69].

Definition 2.16: ([30])

Let $(\Lambda, \sqsubseteq)$ be a finite lattice-ordered index set and let $\{(L_\alpha, \wedge_\alpha, \vee_\alpha)\}_{\alpha\in\Lambda}$ be a Λ-sum family of bounded lattices. Put $L = \bigcup_{\alpha\in\Lambda} L_\alpha$. For every $x \in L$, denote by I_x^{max} and I_x^{min} the set of all maximal and minimal indices of x, respectively and define the binary operations $\wedge$ and $\vee$ on L by:

$$x \wedge y = \begin{cases} x \wedge_\alpha y & if\ (x,y) \in L_\alpha \times L_\alpha, \\ x & if\ (x,y) \in L_\alpha \times L_\beta\ and\ \alpha \sqsubset \beta, \\ y & if\ (x,y) \in L_\alpha \times L_\beta\ and\ \beta \sqsubset \alpha, \\ \top_{\inf\{\alpha^*,\beta^*\}} & if\ x \parallel y, \alpha^* \in I_x^{max}\ and\ , \beta^* \in I_y^{max}. \end{cases} \tag{2.2}$$

and

$$x \vee y = \begin{cases} x \vee_\alpha y & if\ (x,y) \in L_\alpha \times L_\alpha, \\ y & if\ (x,y) \in L_\alpha \times L_\beta\ and\ \alpha \sqsubset \beta, \\ x & if\ (x,y) \in L_\alpha \times L_\beta\ and\ \beta \sqsubset \alpha, \\ \perp_{\sup\{\alpha_*,\beta_*\}} & if\ x \parallel y, \alpha_* \in I_x^{min}\ and\ , \beta_* \in I_y^{min}. \end{cases} \tag{2.3}$$

Then $(L,\wedge,\vee)$ is the lattice-based sum of all $\{(L_\alpha,\wedge_\alpha,\vee_\alpha)\}_{\alpha\in\Lambda}$. This type of lattice-based sum was referred to as *lattice-based sum of bounded lattices*.

Theorem 2.2: ([30])

With all assumptions of Definition 2.16 the lattice-based sum $(L,\wedge,\vee) = \bigoplus_{\alpha\in\Lambda} (L_\alpha,\wedge_\alpha,\vee_\alpha)$ is a bounded lattice.

Remark 2.6:

Given a lattice-based sum $(L,\wedge,\vee) = \bigoplus_{\alpha\in\Lambda} (L_\alpha,\wedge_\alpha,\vee_\alpha)$. The partial order relation $\leq$ on the lattice L obtained by setting $x \leq y$ in L if and only if, $x \wedge y = x$ coincides with the partial order relation given in Definition 2.14. One obtains the same partial order relation from the given lattice by setting $x \leq y$ in L if and only if, $x \vee y = y$.

CHAPTER THREE

ASSOCIATIVE AGGREGATION OPERATORS ON BOUNDED LATTICES

Chapter three
Associative aggregation operators on bounded lattices

3.1 Introduction and preliminaries

The concept of *aggregation* has been introduced in [4, 10, 40] as a process of combining several input values into a single output and the numerical function performing this process is called an *aggregation function* (it is also called *aggregation operator,* both terms are used interchangeably in the existing thesis). Aggregation functions are widely used in pure and applied mathematics, computer and engineering sciences, economics and finance, social science as well as in many other applied fields of physics and natural sciences. Thus, a main characteristic of the aggregation functions is that they are used in a large of areas and disciplines.

If the number of input values to be aggregated is fixed, say n, an aggregation function is a real function of n variables. This is still a too general topic. Therefore, in [4, 10, 40] the considerations regarding inputs as well as outputs are restricted to some fixed interval $[a, b] \subseteq [-\infty, \infty]$, in particular $[0,1]$.

One of the most important classes of aggregation operators on the unit interval is the class of associative aggregation operators. Obviously, there exist multiple associative aggregation operators on the unit interval but the most important and popular ones are the triangular norms, triangular conorms, uninorms and nullnorms (see e.g., [4, 10, 40]).

Stimulating some investigations in topology and logic, associative aggregation operators have been also studied on more general structures such as bounded partially ordered sets and bounded lattices (see e.g., [22, 23, 53, 60, 72]). Therefore, we aim in this chapter to give a survey on the theory of the mentioned associative aggregation operators on bounded lattices.

The general aggregation operator introduced firstly to act on the unit interval in [4, 10, 40] and then on any bounded lattice in [23, 53, 60] as follow

Definition 3.1: ([23, 53, 60])

Let $(L, \leq, \perp, \top)$ be a bounded lattice, and $n \in N$ be fixed. A mapping $A: L^n \to L$ is called an n-ary aggregation function on L whenever it is increasing,

$$A(\mathbf{x}) \leq A(\mathbf{y}) \text{ whenever } \mathbf{x} \leq \mathbf{y} \text{ (i.e. } x_1 \leq y_1, \ldots, x_n \leq y_n \text{)} \quad (3.1)$$

and it satisfies boundary conditions

$$A(\perp, \ldots, \perp) = \perp, A(\top, \ldots, \top) = \top. \quad (3.2)$$

A mapping $B: \bigcup_{n \in N} L^n \to L$ is called an extended aggregation function on L whenever $B \mid L^n$ (B restricted to L^n) is an n-ary aggregation function on L for any $n \in N$.

Remark 3.1:

If $L = [0,1]$ is equipped with the standard ordering of reals, Definition 3.1 turns into the classical definition of an aggregation function on the unit interval [4, 10, 40].

The monotonicity in all arguments and preservation of the bounds in Definition 3.1 are the two fundamental properties that characterize general aggregation operators. If any of these properties fails, we cannot consider the function A as an aggregation operator, because it will provide inconsistent output when used. All other properties leading to useful subclasses of aggregation operators as we can see in Definition 3.2.

Definition 3.2: ([23, 53, 60])

Let A be an aggregation operator on a bounded lattice $(L, \leq, \perp, \top)$,

i) A is said to be ***associative*** if

$$A(x_1, \dots, x_k, \dots, x_n) = A_2\big(A_k(x_1, \dots, x_k), A_{n-k}(x_{k+1}, \dots, x_n)\big)$$

for all $n \geq 2$, $k = 1, \dots, n-1$ and $x_i \in L$ $(i = 1, \dots, n)$.

ii) A is said to be ***commutative*** if

$$A(x_1, \dots, x_n) = A(x_{\pi(1)}, \dots, x_{\pi(n)}).$$

for all $n \in N^+$, $x_i \in L$ $(i = 1, \dots, n)$ and for all permutations $\pi(1), \dots, \pi(n)$ of $\{1, \dots, n\}$

iii) A has a ***neutral element*** $e \in L$ if for all $n \geq 2$ and $x_i \in L$ $(i = 1, \dots, n)$, if $x_k = e$ for some $k \in \{1, \dots, n\}$, then

$$A(x_1, \dots, x_n) = A(x_1, \dots, x_{k-1}, x_{k+1} \dots, x_n).$$

iv) An element $a \in L$ is called a ***zero element*** (***annihilator***) of A if

$\forall x_1, \dots, x_n \in L : a \in \{x_1, \dots, x_n\}$ then $A(x_1, \dots, x_n) = a$.

v) An element $x \in L$ is called an ***idempotent element*** of A whenever $A(x, \dots, x) = x$. Therefore, A is called an ***idempotent aggregation operator*** if each $x \in L$ is an idempotent element of A.

vi) A is called ***conjunctive*** whenever $A(x_1, \dots, x_n) \leq x_i$ for all $i \in \{1, \dots, n\}$

vii) A is called ***disjunctive*** whenever $A(x_1, \dots, x_n) \geq x_i$ for all $i \in \{1, \dots, n\}$

Remark 3.2: ([53])

Note that for any bounded lattice $(L, \leq, \perp, \top)$ a dual bounded lattice $(L^d, \leq^d, \perp^d, \top^d)$ can be introduced, where $L^d = L, x \leq^d y$ if and only if $y \leq x$, and $\perp^d = \top, \top^d = \perp$. Evidently, any aggregation function $A: L^n \to L$ on L can be considered also as an aggregation function on L^d.

Several properties of A on L are the same as those of A on L^d (namely, all algebraic properties not linked to the orderings $\leq$ and $\leq^d$). However, properties based on the ordering should be modified by the above duality (for example, conjunctivity on L is equivalent to the disjunctivity on L^d).

In the following, we will recall the definitions and properties as well as construction methods for the most important associative aggregation operators mentioned earlier on bounded lattices. In the sequel, without loss of generality, we will restrict our consideration on the associative aggregation operator A to two arguments, because due to the associativity, A can be extended to a finite number of arguments. We start by triangular norms and triangular conorms.

3.2 Triangular norms and triangular conorms

3.2.1 Basic definitions and properties

The triangular norms and triangular conorms were introduced by Schweitzer and sklar [68] aiming at an extension of the triangle inequality and following some ideas of Menger [58]. These operators were studied in the framework of many-valued and fuzzy logics in [1, 38, 39, 42, 43]. They were also studied by many authors in other papers [2, 3, 52, 59]. Although the triangular norms and triangular conorms were strictly defined on the unit interval, they were mostly studied on bounded lattices [21, 22, 72].

Definition 3.3: ([67])

Let $(L, \leq, \perp, \top)$ be a bounded lattice. The operation $T: L^2 \to L$ is called a triangular norm (t-norm) if the following conditions are fulfilled for all $x, y, z \in L$:

i. $T(x,y) = T(y,x)$ (Commutativity)

ii. $T\big(x, T(y,z)\big) = T(T(x,y), z)$ (Associativity)

iii. $T(x,z) \leq T(y,z)$ whenever $x \leq y$ (Monotonicity)

iv. $T(x,\top) = x$ (Neutral element)

Definition 3.4: ([34])

Let $(L, \leq, \perp, \top)$ be a bounded lattice. The operation $S: L^2 \to L$ is called a triangular conorm (t-conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $\perp \in L$.

Example 3.1:

There exist at least two t-norms and two t-conorms acting on any bounded lattice L:

- The minimum $T_M^L: L^2 \to L, T_M^L(x,y) = x \wedge y$.
- The drastic product $T_D^L: L^2 \to L$,

$$T_D^L(x,y) = \begin{cases} x \wedge y & if\ \top \in \{x,y\}, \\ \perp & otherwise. \end{cases}$$

- The maximum $S_M^L: L^2 \to L, S_M^L(x,y) = x \vee y$.
- The drastic sum $S_D^L: L^2 \to L$,

$$S_D^L(x,y) = \begin{cases} x \vee y & if\ \perp \in \{x,y\}, \\ \top & otherwise. \end{cases}$$

Main properties:

The t-norm and t-conorm operations introduced in Definition 3.3 and Definition 3.4, respectively, have the following properties on any bounded lattice $(L, \leq, \perp, \top)$

i) For any t-norm T and any t-conorm S on L, the following additional boundary conditions are satisfied

$$T(x, \perp) = T(\perp, x) = \perp, \qquad S(x, \top) = S(\top, x) = \top$$

It means that $\perp \in L$ is acting as the zero element of T and $\top \in L$ is acting as the zero element of S.

ii) If, for two t-norms T_1 and T_2, the inequality $T_1(x, y) \leq T_2(x, y)$ holds for all $(x, y) \in L^2$, then we say that T_1 is *weaker than* T_2 (equivalent to T_2 is *stronger than* T_1) and we write $T_1 \leq T_2$. Similarly, in the t-conorm case if $S_1 \leq S_2$, then we say that S_1 is *weaker than* S_2 (equivalent to S_2 is *stronger than* S_1).

iii) Due to the monotonicity of T, then for each t-norm T, and for each $(x, y) \in L^2$ we have both $T(x, y) \leq T(x, \top) = x$ and $T(x, y) \leq T(\top, y) = y$. Also, for all $(x, y) \in L \backslash \{\perp, \top\}$ we trivially have $T(x, y) \geq \perp = T_D^L(x, y)$ and hence, we have

$$T_D^L \leq T \leq T_M^L$$

It means that, the drastic product t-norm is the *weakest* t-norm and the minimum t-norm is the *strongest* one. In a similar way and by using the duality, in the t-conorm case, we have

$$S_M^L \leq S \leq S_D^L$$

The only idempotent t-norm T on L is the minimum T_M^L and the only idempotent t-conorm S on L is the maximum S_M^L.

Remark 3.3:

Note that, if $L = [0,1]$ (i.e., L is the classical unit interval), then we have the following:

i) There exist uncountable many t-norms and t-conorms acting on [0,1]. However, the following are the four basic t-norms and t-conorms acting on [0,1] extracted from [52]

- $T_M(x, y) = \min(x, y)$, (Minimum)
- $T_P(x, y) = x.y$, (Product)
- $T_L(x, y) = \max(x + y - 1, 0)$, (Lukasiewiz t-norm)
- $T_D(x, y) = \begin{cases} 0 & if\ (x, y) \in [0,1[^2, \\ \min(x, y) & otherwise \end{cases}$ (Drastic product)
- $S_M(x, y) = \max(x, y)$, (Maximum)
- $S_P(x, y) = x + y - x.y$ (Probabilistic sum)
- $S_L(x, y) = \min(x + y, 1)$ (Lukasiewiz t-conorm)
- $S_D(x, y) = \begin{cases} 1 & if\ (x, y) \in]0,1]^2, \\ \max(x, y) & otherwise. \end{cases}$ (Drastic sum)

ii) Due to monotonicity of T and S, we have the following order for the four basic t-norms and t-conorms on [0,1]

$$T_D \le T_L \le T_P \le T_M, \quad S_M \le S_P \le S_L \le S_D$$

3.2.2 Construction methods

There are many ways for constructing t-norms and t-conorms from given ones on the unit interval such as ***Pseudo-inverse of monotone functions***, ***additive and multiplicative generators*** and ***ordinal sums*** [52]. The latter is the most important one for this purpose. Since t-norms are special compact semigroups (i.e., t-norms are binary and associative functions with neutral element 1), the concept of ordinal sums in the sense of Clifford [17] provided a method to construct new t-norms from given ones (similarly, for t-conorms by duality). There are several papers concerning ordinal sums of t-norms (t-conorms) on the unit interval, see e.g. [46, 47, 51, 52]. Stimulating some investigation in topology and logic, the ordinal sum construction was generalized on a general bounded lattices, see e.g. [15, 34, 56, 57, 65-67], inspired first of all by Goguen's proposal to consider fuzzy sets with membership values from a bounded lattices. In the following we will recall all attempts for constructing t-norms and t-conorms on bounded lattices via the ordinal sum method.

Definition 3.5: ([66, 67])

Given a bounded lattice $(L, \leq, \perp, \top)$, a linearly ordered index set $(I, \preccurlyeq_I)$, a family of pairwise disjointed subintervals of L, $\{]a_i, b_i[\}_{i\in I}$ and a family of t-norms $\{T^{[a_i,b_i]}\}_{i\in I}$ on the corresponding intervals $\{[a_i, b_i]\}_{i\in I}$. The operation $T: L^2 \to L$ defined as follows:

$$T(x, y) = \begin{cases} T^{[a_i,b_i]}(x, y) & if\ (x, y) \in [a_i, b_i]^2, \\ x \wedge y & otherwie. \end{cases} \tag{3.3}$$

is called the *ordinal sum* of the family $\{T^{[a_i,b_i]}\}_{i\in I}$on L.

By duality, we can define the ordinal sum of t-conorms on bounded lattices in the following way

Definition 3.6: ([66, 67])

Given a bounded lattice $(L, \leq, \perp, \top)$, a linearly ordered index set $(I, \preccurlyeq_I)$, a family of pairwise disjointed subintervals of L, $\{]a_i, b_i[\}_{i\in I}$ and a family of t-conorms $\{S^{[a_i,b_i]}\}_{i\in I}$ on the corresponding intervals $\{[a_i, b_i]\}_{i\in I}$. The operation $S: L^2 \to L$ defined as follow

$$S(x,y) = \begin{cases} S^{[a_i,b_i]}(x,y) & if\ (x,y) \in [a_i, b_i]^2, \\ x \vee y & otherwie. \end{cases} \quad (3.4)$$

is called the *ordinal sum* of the family $\{S^{[a_i,b_i]}\}_{i\in I}$ on L.

Note that, if $L = [0,1]$ (L is the classical unit interval) then the ordinal sums T and S defined in Equations (3.3) and (3.4), respectively, are reduced to the ordinal sum of t-norms and t-conorms on the unit interval [52]. It has been shown in [52], that the ordinal sums T and S are t-norm and t-conorm for any family of t-norms and t-conorms on the unit interval, but as shown in [67], [34], the ordinal sums T and S in Definition 3.5 and Definition 3.6 are not a t-norm and a t-conorm on a general bounded lattice L, respectively. This can be seen in the following example for ordinal sums T and S of one summand only.

Example 3.2:

Consider the bounded lattice $(L, \leq, \perp, \top)$ in Figure 3-1, a subintervals $[b, \top] = \{b, c, \top\}$ and $[\perp, b] = \{\perp, d, b]$. The ordinal sum of the t-norm $T_D^{[b,\top]}$ is the operator T defined by Equation (3.3) which values are written in Table 3-1. Also, the ordinal sum of the t-conorm $S_D^{[\perp,b]}$ is the operator S defined by Equation (3.4) which values are written in Table 3-2. The ordinal sums T and S described above are not a t-norm and a t-conorm on L, respectively, such that, if we consider $a, c \in L$, then we have

$$T(T(c,c),a) = T\left(T_D^{[b,\top]}(c,c),a\right) = T(b,a) = b \wedge a = d,$$
$$T\big(c,T(c,a)\big) = T(c,c \wedge a) = T(c,a) = c \wedge a = a.$$

Since $d \neq a$, T is not associative. Also, it is easy to see that $a \leq c$, but we have

$$T(a,c) = a \wedge c = a,\ \ T(c,c) = T_D^{[b,\top]}(c,c) = b$$

Since $a \parallel b$ ($T(a,c) \parallel T(c,c)$ for $a \leq c$), T is not monotone.

Similarly, if we consider $a,d \in L$, then we have

$$S(S(d,d),a) = S\left(S_D^{[\perp,b]}(d,d),a\right) = S(b,a) = b \vee a = c,$$
$$S\big(d,S(d,a)\big) = S(d,d \vee a) = S(d,a) = d \vee a = d.$$

Since $c \neq d$, S is not associative. Also, it is easy to see that $d \leq a$, but we have

$$S(d,d) = S_D^{[\perp,b]}(d,d) = b,\ \ S(d,a) = d \vee a = a$$

Since $b \parallel a$, S is not monotone.

To save time and effort to test the associativity of T (similarly, for S) of Example 3.2, the python code in appendix A can be used which give the output "`(False, (a, c, c))`" to indicate that T is not associative

i.e. $T\big(a,T(c,c)\big) \neq T(T(a,c),c)$

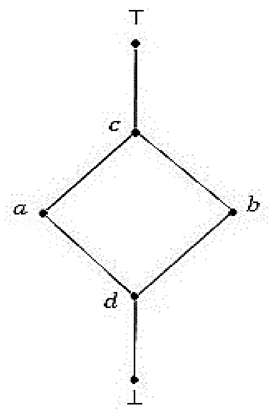

Figure 3-1 The lattice L of Example 3.2

Table 3-1 The operation T on L of Example 3.2

T	$\bot$	d	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
d	$\bot$	d	d	d	d	d
a	$\bot$	d	a	d	a	a
b	$\bot$	d	d	b	b	b
c	$\bot$	d	a	b	b	c
$\top$	$\bot$	d	a	b	c	$\top$

Table 3-2 The operation S on L of Example 3.2

S	$\bot$	d	a	b	c	$\top$
$\bot$	$\bot$	d	a	b	c	$\top$
d	d	b	a	b	c	$\top$
a	a	a	a	c	c	$\top$
b	b	b	c	b	c	$\top$
c	c	c	c	c	c	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

As we can see in Example 3.2, the ordinal sum construction method may not work to construct t-norms and t-conorms from given ones on a general bounded lattice.

Note that, as shown in [65, 67], if the underlying bounded lattice L is describable as ordinal sum of intervals, then for any family of t-norms (t-conorms) on that intervals, the resultant ordinal sums T (S) will be a t-norm (a t-conorm) on L. But the converse isn’t true in general, such that (see Example 4.2 in [67]), there exist ordinal sum t-norms on a bounded lattice L, although L isn’t an ordinal sum of intervals.

Therefore, for necessary and sufficient conditions to be met to ensure that the ordinal sums T and S in Equations (3.3) and (3.4) are, respectively a t-norm and a t-conorm on a general bounded lattice, the authors in [65, 67] have presented a good discussion. These conditions are recalled in Theorem 3.1 for ordinal sum T only. The same results for S can be obtained by duality.

Theorem 3.1: ([65, 67])

Consider some bounded lattice $(L, \leq, \perp, \top)$, some index set I, and a family of pairwise disjoint subintervals $\{]a_i, b_i[\}_{i \in I}$ of L. Then the following are equivalent:

i) The ordinal sum $T: L^2 \rightarrow L$ defined by Equation (3.3) is a t-norm for arbitrary $T^{[a_i, b_i]}$ on $[a_i, b_i]$.

ii) For all $x \in L$ and for all $i \in I$ it holds that

a) If x is incomparable to a_i, then it is incomparable to all $u \in [a_i, b_i[$.

b) If x is incomparable to b_i, then it is incomparable to all $u \in]a_i, b_i]$.

Theorem 3.1 states that, if the underlying bounded lattice L is describable as ordinal or horizontal sum of chains, then for any family of t-norms (t-conorms) on such bounded lattice, the ordinal sums T and S in Equations (3.3) and (3.4) are, respectively, a t-norm and a t-conorm on L.

Example 3.3:

Consider the bounded lattice $(L, \leq, \perp, \top)$ shown in Figure 3-1, a subinterval $[d, c] = \{d, a, b, c\}$. Then, for any t-norm $T^{[d,c]}$ on $[d, c]$, T defined by Equation (3.3) is a t-norm on L. Also, for any t-conorm $S^{[d,c]}$ on $[d,c]$, S defined by Equation (3.4) is a t-conorm on L. The main reason is that, L is describable as ordinal sum of intervals, i.e. $L = [\bot, d] \oplus [d,c] \oplus [c, \top]$.

It is worth to be mentioned that, the conditions given in Theorem 3.1 do not seem very efficient to be used in order to prove whether a *specific* ordinal sum T, with respect to a *particular* family of t-norms, is a t-norm, because the ordinal sum T might be a t-norm even when is not a t-norm for any family of t-norms, as we can see in the following example which is extracted from [57].

Example 3.4:

Consider the lattice $(L, \leq, \bot, \top)$ in Figure 3-2 and the ordinal sum T of the t-norm $T^{[\bot,b]}$, given by Equation (3.3), whose values are written in Table 3-3. We see that $[\bot, b]$ does not satisfy the conditions given in Theorem 3.1(ii), since x and b are incomparable, $c \leq x$ and $c \in]\bot, b]$. Although, T is a t-norm, which can be easily checked.

Again, we can use the python code in appendix B for testing the associativity of T of Example 3.4 which give the output "`(True, None)`" to indicate that T is associative in all cases.

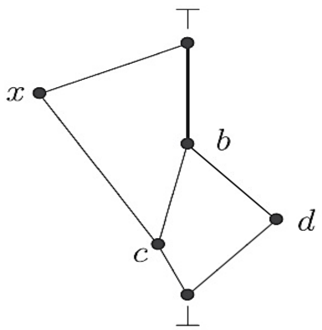

Figure 3-2 The lattice L of Example 3.4

Table 3-3 The t-norm T on L of Example 3.4

T	$\perp$	c	d	b	x	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
c	$\perp$	c	$\perp$	c	c	c
d	$\perp$	$\perp$	$\perp$	d	$\perp$	d
b	$\perp$	c	d	b	c	b
x	$\perp$	c	$\perp$	c	x	x
$\top$	$\perp$	c	d	b	x	$\top$

[56] and [57] gave extra necessary and sufficient conditions to ensure that an ordinal sum of t-norms is a t-norm on bounded lattices. It turned out from [56] that, to check if an ordinal sum is a t-norm for any family of t-norms, we only need to consider on each subinterval a drastic t-norm (which is the simplest t-norm) and to verify if the new ordinal sum is a t-norm.

Now, we need to recall all other attempts for constructing t-norms and t-conorms on bounded lattices via ordinal sum technique. We start by the constructions given in [34].

Theorem 3.2: ([34])

Let $(L, \leq, \perp, \top)$ be a bounded lattice and let $a \in L \backslash \{\perp, \top\}$. If V is a t-norm on $[a, \top]$ and W is a t-conorm on $[\perp, a]$, then the functions $T_1: L^2 \to L$ and $S_1: L^2 \to L$ are, respectively, a t-norm and a t-conorm on L, where

$$T_1(x,y) = \begin{cases} V(x,y) & if\ x, y \in [a, \top[, \\ x \wedge y & if\ \top \in \{x, y\}, \\ x \wedge y \wedge a & otherwise. \end{cases} \tag{3.5}$$

and

$$S_1(x,y)=\begin{cases} W(x,y) & if\ x,y\in]\perp,a], \\ x\vee y & if\ \ \perp\in\{x,y\}, \\ x\vee y\vee a & otherwise. \end{cases} \tag{3.6}$$

In [15] another construction methods for t-norms and t-conorms from given ones on bounded lattices have been presented. These constructions are also based on one starting t-norm V and one starting t-conorm W but it is different from the constructions methods in Theorem 3.2, as we can see in Theorem 3.3

Theorem 3.3: ([15])

Let $(L,\leq,\perp,\top)$ be a bounded lattice and let $a\in L\backslash\{\perp,\top\}$. If V is a t-norm on $[a,\top]$ and W is a t-conorm on $[\perp,a]$, then the functions $T_2: L^2\rightarrow L$ and $S_2: L^2\rightarrow L$ are, respectively, a t-norm and a t-conorm on L, where

$$T_2(x,y)=\begin{cases} V(x,y) & if\ x,y\in[a,\top[, \\ x\wedge y & if\ \top\in\{x,y\}, \\ \perp & otherwise. \end{cases} \tag{3.7}$$

and

$$S_2(x,y)=\begin{cases} W(x,y) & if\ x,y\in]\perp,a], \\ x\vee y & if\ \ \perp\in\{x,y\}, \\ \top & otherwise. \end{cases} \tag{3.8}$$

Example 3.5:

Consider the bounded lattice $(L,\leq,\perp,\top)$ in Figure 3-1. Let $T^{[b,\top]}=T_D^{[b,\top]}$ and $S^{[\perp,b]}=S_D^{[\perp,b]}$. It is easy to check that the function T_1 whose values are written in Table 3-4 is a t-norm on L for the t-norm $T^{[b,\top]}$ using Equation (3.5) and the function S_1 whose values are written in Table 3-5 is a t-conorm on L for the t-conorm $S^{[\perp,b]}$ using

Equation (3.6). Also, the function T_2 whose values are written in Table 3-6 is a t-norm on L for the t-norm $T^{[b,\top]}$ using Equation (3.7) and the function S_2 whose values are written in Table 3-7 is a t-conorm on L for the t-conorm $S^{[\perp,b]}$ using Equation (3.8).

Table 3-4 The t-norm T_1 on L of Example 3.5

T_1	$\perp$	d	a	b	c	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
d	$\perp$	d	d	d	d	d
a	$\perp$	d	d	d	d	a
b	$\perp$	d	d	b	b	b
c	$\perp$	d	d	b	b	c
$\top$	$\perp$	d	a	b	c	$\top$

Table 3-5 The t-conorm S_1 on L of Example 3.5

S_1	$\perp$	d	a	b	c	$\top$
$\perp$	$\perp$	d	a	b	c	$\top$
d	d	b	a	b	c	$\top$
a	a	a	a	c	c	$\top$
b	b	b	c	b	c	$\top$
c	c	c	c	c	c	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Table 3-6 The t-norm T_2 on L of Example 3.6

T_2	$\perp$	d	a	b	c	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
d	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	d
a	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	a
b	$\perp$	$\perp$	$\perp$	b	b	b
c	$\perp$	$\perp$	$\perp$	b	b	c
$\top$	$\perp$	d	a	b	c	$\top$

Table 3-7 The t-conorm S_2 on L of Example 3.6

S_2	$\perp$	d	a	b	c	$\top$
$\perp$	$\perp$	d	a	b	c	$\top$
d	d	b	$\top$	b	$\top$	$\top$
a	a	$\top$	$\top$	$\top$	$\top$	$\top$
b	b	b	$\top$	b	$\top$	$\top$
c	c	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Remark 3.4:

Given a bounded lattice $(L, \leq, \perp, \top)$. The operations T_1, T_2, S_1 and S_2 just described in Theorems 3.2 and 3.3, are based on one starting t-norm V acting on a subinterval $[a, \top]$ in the t-norm case, and one starting t-conorm W acting on a subinterval $[\perp, a]$ in the t-conorm case. That is, we cannot force T_1, T_2, S_1 and S_2 to coincide with a predescribed t-norm H on $[\perp, a]$ and a t-conorm H on $[a, \top]$ and expect that T_1 and T_2 are still a t-norm and S_1 and S_2 a t-conorm on L, respectively.

3.3 Uninorms

3.3.1 Basic definitions and properties

Uninorms on the unit interval are one of the most important associative aggregation operators with neutral element $e \in [0,1]$ that generalize t-norms and t-conorms operators. This generalization stems from the location of the neutral element, such that, in the uninorm case, the neutral element is any element laying anywhere on the unit interval rather than at 1 as in the t-norm case or at 0 as in the t-conorm case. This operators have been firstly introduced in [71]. They were also studied on the unit interval by many authors in other papers, for example, in [18, 20, 24, 27-29, 36, 44, 63, 64]. Recently, this operators have been introduced on bounded lattices in [50], showing the existence of uninorms on an arbitrary bounded lattice L with the neutral element e laying anywhere in the bounded lattice L, using the fact that the t-norms and t-conorms on arbitrary bounded lattice L always exist. Our interest in the construction of these operations requires us to mention that there were several methods for constructing uninorms on bounded lattices introduced in [7, 11, 14, 50]. We will recall all of these constructions after some concepts and properties concerning uninorms on bounded lattices.

Definition 3.7: ([50])

Let $(L, \leq, \perp, \top)$ be a bounded lattice. Operation $U: L^2 \to L$ is called a *uninorm* on L if it is commutative, associative, increasing with respect to both variables and has a neutral element $e \in L$.

It is clear form Definition 3.7 that the t-norm and the t-conorm operators are special cases of uninorm operator, such that, if $e = \top$, then it is the case of t-norm, also if $e = \perp$, then it is the case of t-conorm.

Definition 3.8: ([11])

Let $(L, \leq, \perp, \top)$ be a bounded lattice, $e \in L \backslash \{\perp, \top\}$ and U a uninorm on L with the neutral element e.

i) U is called a *conjunctive* uninorm if $U(\perp, \top) = \perp$.

ii) U is called a *disjunctive* uninorm if $U(\perp, \top) = \top$.

Proposition 3.1: ([50])

Let $(L, \leq, \perp, \top)$ be a bounded lattice, $e \in L \backslash \{\perp, \top\}$ and U a uninorm on L with the neutral element e, then

i) $T_U = U|_{[\perp, e]^2}: [\perp, e]^2 \to [\perp, e]$ is a t-norm on $[\perp, e]$.

ii) $S_U = U|_{[e, \top]^2}: [e, \top]^2 \to [e, \top]$ is a t-conorm on $[e, \top]$.

Proposition 3.2: ([50])

Let $(L, \leq, \perp, \top)$ be a bounded lattice, and let $e \in L \backslash \{\perp, \top\}$ and U a uninorm on L with the neutral element e, then the following hold:

i) $x \wedge y \leq U(x, y) \leq x \vee y \; \forall (x, y) \in [\perp, e] \times [e, \top] \cup [e, \top] \times [\perp, e]$.

ii) $U(x, y) \leq x \; \forall (x, y) \in L \times [\perp, e]$.

iii) $U(x, y) \leq y \; \forall (x, y) \in [\perp, e] \times L$.

iv) $x \leq U(x, y) \; \forall (x, y) \in L \times [e, \top]$.

v) $y \leq U(x,y)\ \forall (x,y) \in [e,\top] \times L$.

3.3.2 Construction methods

On the unit interval, (see Figure 3-3), a uninorm U with neutral element e is acting as a t-norm on $[0,e]^2$ and a t-conorm on $[e,1]^2$ while on the remaining parts of the unit square, a uninorm is acting as averaging aggregation function between the minimum and maximum operators. It means that, we can construct a uninorm U on the unit interval $[0,1]$ by means of any t-norm T and any t-conorm S acting on $[0,1]$ just describe U on the rest of the unit square, for example, for any t-norm T and any t-conorm S on $[0,1]$, if we consider that $U(x,y) = \min(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$, then we obtain a uninorm U belonging to the general class of minimum uninorms denoted by U_{min} (see Figure 3-4 (a)). Similarly, if $U(x,y) = \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$ then we obtain a uninorm U belonging to the general class of maximum uninorms denoted by U_{max} (see Figure 3-4 (b)).

In the case of bounded lattice $(L,\leq,\perp,\top)$, we may have one or more elements incomparable with e and hence the characterization of uninorms on bounded lattices is different from given ones on the unit intervals. In [49] a characterization of uninorms on bounded lattices has been introduced by means of a t-norm T, a t-conorm S and four symmetric aggregation functions H_1, H_2, H_3 and H_4 (see Figure 3-5). But, as shown in [49], recalling the problems with constructing triangular norms (conorms) on a bounded lattice by means of ordinal sum approach, it is not surprising that we are not able to ensure the existence of a proper uninorm U acting on a bounded lattice L, with a neutral element $e \in L \setminus \{\perp, \top\}$. There exist several attempts to construct uninorms on bounded lattices. We start by two constructions given in [50].

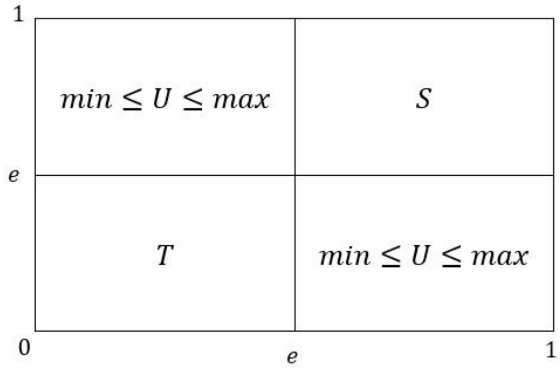

Figure 3-3 The structure of uninorm on [0,1]

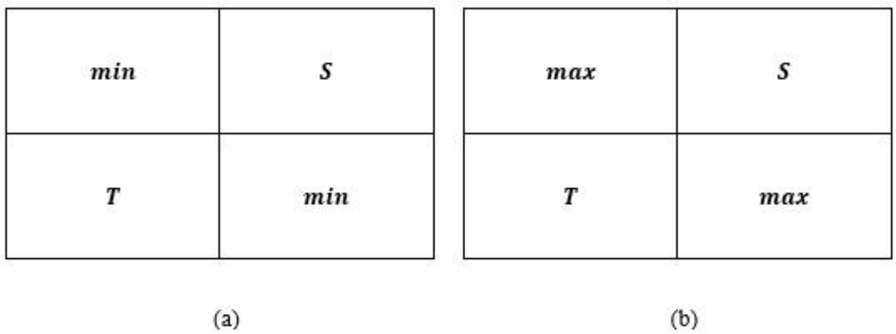

Figure 3-4 (a) A member of U_{min} , (b) A member of U_{max}

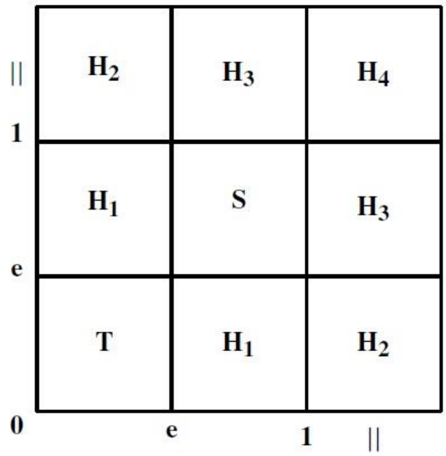

Figure 3-5 The structure of uninorms on bounded lattices

Theorem 3.4: ([50])

Let $(L, \leq, \perp, \top)$ be a bounded lattice and let $e \in L \backslash \{\perp, \top\}$. If T_e is a t-norm on $[\perp, e]^2$ and S_e is a t-conorm on $[e, \top]^2$, then the functions $U_{T_1}: L^2 \to L$ and $U_{S_1}: L^2 \to L$ defined as follow

$$U_{T_1}(x, y) = \begin{cases} T_e(x, y) & if\ (x, y) \in [\perp, e]^2, \\ x \vee y & if\ (x, y) \in [\perp, e] \times (e, \top] \cup (e, \top] \times [\perp, e], \\ y & if\ x \in [\perp, e], y \parallel e, \\ x & if\ y \in [\perp, e], x \parallel e, \\ \top & otherwise. \end{cases} \tag{3.9}$$

and

$$U_{S_1}(x, y) = \begin{cases} S_e(x, y) & if\ (x, y) \in [e, \top]^2, \\ x \wedge y & if\ (x, y) \in [\perp, e) \times [e, \top] \cup [e, \top] \times [\perp, e), \\ y & if\ x \in [e, \top], y \parallel e, \\ x & if\ y \in [e, \top], x \parallel e, \\ \perp & otherwise. \end{cases} \tag{3.10}$$

are uninorms on L with neutral element e.

Another two construction methods have been introduced in [11] that are different from the proposal given in [50]. We recall these constructions in Theorem 3.5.

Theorem 3.5: ([11])

Let $(L, \leq, \perp, \top)$ be a bounded lattice and let $e \in L \setminus \{\perp, \top\}$. If T_e is a t-norm on $[\perp, e]^2$ and S_e is a t-conorm on $[e, \top]^2$, then the functions $U_{T_2}: L^2 \to L$ and $U_{S_2}: L^2 \to L$ defined as

$$U_{T_2}(x,y) = \begin{cases} T_e(x,y) & if\ (x,y) \in [\perp, e]^2, \\ y & if\ x \in [\perp, e], y \parallel e, \\ x & if\ y \in [\perp, e], x \parallel e, \\ x \vee y & otherwise. \end{cases} \tag{3.11}$$

and

$$U_{S_2}(x,y) = \begin{cases} S_e(x,y) & if\ (x,y) \in [e, \top]^2, \\ y & if\ x \in [e, \top], y \parallel e, \\ x & if\ y \in [e, \top], x \parallel e, \\ x \wedge y & otherwise. \end{cases} \tag{3.12}$$

are uninorms on L with neutral element e.

Remark 3.5:

Given a bounded lattice $(L, \leq, \perp, \top)$, then

i) The uninorms U_{T_1} and U_{S_1} cannot be used for constructing idempotent uninorms on L, such that, if we consider the only idempotent t-norm T_M^L on $[\perp, e]$, then the corresponding uninorm U_{T_1} is not an idempotent uninorm having value $\top$ on the domain $(L \setminus [\perp, e]^2)$. The case of U_{S_1} is similar. However, the uninorms U_{T_2} and U_{S_2} can be applied to show the existence of idempotent uninorms on L for any $e \in L \setminus \{\perp, \top\}$.

ii) The uninorms U_{T_2} and U_{S_2} in Theorem 3.5 can be equivalently defined by

$$U_{T_2}(x,y) = \begin{cases} T_e(x,y) & if\ (x,y) \in [\perp, e]^2, \\ H(x) \vee H(y) & otherwise. \end{cases}$$

$$U_{S_2}(x,y) = \begin{cases} S_e(x,y) & if\ (x,y) \in [e,\top]^2, \\ M(x) \wedge M(y) & otherwise. \end{cases}$$

where $H, M: L \to L$ are mappings given by

$$H(x) = \begin{cases} \perp & if\ x \in [\perp, e], \\ x & otherwise. \end{cases}$$

$$M(x) = \begin{cases} \top & if\ x \in [e, \top], \\ x & otherwise. \end{cases}$$

Example 3.7:

Consider the bounded lattice $(L, \leq, \perp, \top)$ in Figure 3-6. Let $T_e = T_D^{[\perp,e]}$ on $[\perp, e]$ and $S_e = S_D^{[e,\top]}$ on $[e,\top]$. Then the operations $U_{T_1}, U_{S_1}, U_{T_2}$and U_{S_2} whose values are written in Tables 3-8, 3-9, 3-10 and 3-11, respectively, are uninorms on L which are constructed using Equations (3.9), (3.10), (3.11) and (3.12), respectively.

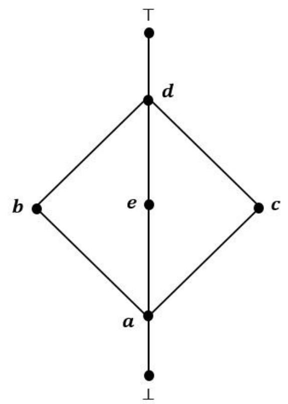

Figure 3-6 The lattice L of Example 3.7

Table 3-8 The uninorm U_{T_1} on L of Example 3.7

U_{T_1}	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	b	c	d	$\bot$	$\top$
a	$\bot$	$\bot$	b	c	d	a	$\top$
b	b	b	$\top$	$\top$	$\top$	b	$\top$
c	c	c	$\top$	$\top$	$\top$	c	$\top$
d	d	d	$\top$	$\top$	$\top$	d	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Table 3-9 The uninorm U_{S_1} on L of Example 3.7

U_{S_1}	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	a	a	a
b	$\bot$	$\bot$	$\bot$	$\bot$	b	b	b
c	$\bot$	$\bot$	$\bot$	$\bot$	c	c	c
d	$\bot$	a	b	c	$\top$	d	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\bot$	a	b	c	$\top$	$\top$	$\top$

Table 3-10 The uninorm U_{T_2} on L of Example 3.7

U_{T_2}	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	b	c	d	$\bot$	$\top$
a	$\bot$	$\bot$	b	c	d	a	$\top$
b	b	b	b	d	d	b	$\top$
c	c	c	d	c	d	c	$\top$
d	d	d	d	d	d	d	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Table 3-11 The uninorm U_{S_2} on L of Example 3.7

U_{S_2}	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	a	a	a	a
b	$\bot$	a	b	a	b	b	b
c	$\bot$	a	a	c	c	c	c
d	$\bot$	a	b	c	$\top$	d	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\bot$	a	b	c	$\top$	$\top$	$\top$

In [14], another two construction methods yielding uninorms on bounded lattices have been presented but with some additional constraints on the neutral element $e \in L\backslash\{\bot, \top\}$ as we can see in Theorem 3.6

Theorem 3.6: ([14])

Let $(L, \leq, \bot, \top)$ be a bounded lattice and fix $e \in L\backslash\{\bot, \top\}$. Suppose that $x \vee y > e$ for all $x \parallel e$ and $y \parallel e$ or $x \vee y \parallel e$ for all $x \parallel e$ and $y \parallel e$. If T_e is a t-norm on $[\bot, e]$, then the function $U_{T_3}: L^2 \to L$ defined as

$$U_{T_3}(x,y)=\begin{cases} T_e(x,y) & if\ (x,y)\in[\bot,e]^2,\\ x\vee y & if\ (x,y)\in A(e)\cup N_e\times N_e,\\ y & if\ (x,y)\in[\bot,e]\times N_e,\\ x & if\ (x,y)\in N_e\times[\bot,e],\\ \top & otherwise.\end{cases} \tag{3.13}$$

is a uninorm on L with neutral element e, where

$$A(e)=[\bot,e]\times[e,\top]\cup[e,\top]\times[\bot,e],\ \ N_e=\{x\in L|x\parallel e\}$$

Theorem 3.7: ([14])

Let $(L,\leq,\bot,\top)$ be a bounded lattice and fix $e\in L\backslash\{\bot,\top\}$. Suppose that $x\wedge y<e$ for all $x\parallel e$ and $y\parallel e$ or $x\wedge y\parallel e$ for all $x\parallel e$ and $y\parallel e$. If S_e is a t-conorm on $[e,\top]$, then the function $U_{S_3}: L^2\to L$ defined as

$$U_{S_3}(x,y)=\begin{cases} S_e(x,y) & if\ (x,y)\in[e,\top]^2,\\ x\wedge y & if\ (x,y)\in A(e)\cup N_e\times N_e,\\ y & if\ (x,y)\in[e,\top]\times N_e,\\ x & if\ (x,y)\in N_e\times[e,\top],\\ \bot & otherwise.\end{cases} \tag{3.14}$$

is a uninorm on L with neutral element e, where $A(e)$ and N_e are as described in Theorem 3.6.

Example 3.8:

i) The bounded lattice L in Figure 3-6 is a positive example satisfying constraints of Theorems 3.6 and 3.7, since $b\vee c=d>e$ for $b\parallel e$ and $c\parallel e$.

ii) The bounded lattice L in Figure 3-7 is a negative example of Theorem 3.6, where, for a chosen neutral element e, constraints of Theorem 3.6 are violated such that $x\vee z=k>e$ for $x\parallel e$ and $z\parallel e$ while, $y\vee m=m\parallel e$ for $y\parallel e$ and $m\parallel e$.

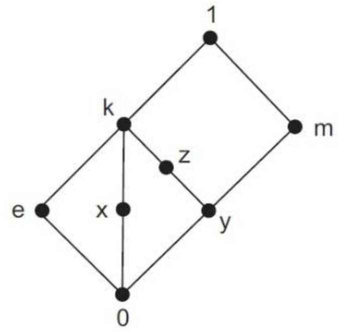

Figure 3-7 The lattice L of Example 3.8 (ii)

iii) Consider the bounded lattice $(L, \leq, \bot, \top)$ in Figure 3-6 and let $T_e = T_D^{[\bot,e]}$ on $[\bot, e]$ and $S_e = S_D^{[e,\top]}$ on $[e, \top]$. Then the operations U_{T_3} and U_{S_3} whose values are written in Tables 3-12 and 3-13, respectively, are uninorms on L which are constructed using Equations (3.13) and (3.14), respectively.

Table 3-12 The uninorm U_{T_3} on L of Example 3.8 (iii)

U_{T_3}	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	b	c	d	$\bot$	$\top$
a	$\bot$	$\bot$	b	c	d	a	$\top$
b	b	b	$\top$	d	$\top$	b	$\top$
c	c	c	d	$\top$	$\top$	c	$\top$
d	d	d	$\top$	$\top$	$\top$	d	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Table 3-13 The uninorm U_{S_3} on L of Example 3.8 (iii)

U_{S_3}	$\perp$	a	b	c	d	e	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	$\perp$	$\perp$	$\perp$	a	a	a
b	$\perp$	$\perp$	$\perp$	a	b	b	b
c	$\perp$	$\perp$	a	$\perp$	c	c	c
d	$\perp$	a	b	c	$\top$	d	$\top$
e	$\perp$	a	b	c	d	e	$\top$
$\top$	$\perp$	a	b	c	$\top$	$\top$	$\top$

3.4 Nullnorms

3.4.1 Basic definitions and properties

Nullnorms on the unit interval with zero element $a \in [0,1]$ are other associative aggregation functions that generalize both t-norms and t-conorms with the opposite behavior of uninorm, such that, they are acting as t-conorms on $[0, a]^2$ and as t-norms on $[a, 1]^2$. These operators have been firstly introduced on the unit interval in [54] and [9]. In the literature, there are some other papers about nullnorms on the real unit interval, for example, [25, 26, 55, 70].

These operators have been introduced on bounded lattices in [48], showing the existence of nullnorms on bounded lattices with zero element $a \in L\backslash\{\perp, \top\}$ using the fact that some t-norms and t-conorms on an arbitrary bounded lattice L always exist.

Our interest in the construction of these operations requires us to mention that there were several methods for constructing nullnorms on bounded lattices introduced in [12, 13, 16, 35, 45, 48]. We will recall

all of these constructions after some concepts concerning nullnorms on bounded lattices.

Definition 3.9: ([48])

Let $(L, \leq, \perp, \top)$ be a bounded lattice. A commutative, associative, non-decreasing in each argument function $V: L^2 \to L$ is called a *nullnorm* if there is an element $a \in L$ such that $V(x, \perp) = x$ for all $x \leq a$ and $V(x, \top) = x$ for all $x \geq a$.

It can be easily derived that $V(x, a) = a$ for all $x \in L$. Thus, a is the zero element of V.

Proposition 3.3: ([48])

Let $(L, \leq, \perp, \top)$ be a bounded lattice, $a \in L \backslash \{\perp, \top\}$ and V a nullnorm on L with zero element a. Then

i) $S_V = V|_{[\perp,a]^2}: [\perp, a]^2 \to [\perp, a]$ is a t-conorm on $[\perp, a]$.

ii) $T_V = V|_{[a,\top]^2}: [a, \top]^2 \to [a, \top]$ is a t-norm on $[a, \top]$.

Proposition 3.4: ([48])

Let $(L, \leq, \perp, \top)$ be a bounded lattice, $a \in L \backslash \{\perp, \top\}$ and V a nullnorm on L with zero element a. Then the following hold

i) $V(x, y) = a \ \forall (x, y) \in [\perp, a] \times [a, \top] \cup [a, \top] \times [\perp, a]$.

ii) $a \leq V(x, y) \ \forall (x, y) \in [a, \top]^2 \cup [a, \top] \times N_a \cup N_a \times [a, \top]$.

iii) $V(x, y) \leq a \ \forall (x, y) \in [\perp, a]^2 \cup [\perp, a] \times N_a \cup N_a \times [\perp, a]$.

iv) $V(x, y) \leq y \ \forall (x, y) \in L \times [a, \top]$.

v) $V(x, y) \leq x \ \forall (x, y) \in [a, \top] \times L$.

vi) $x \leq V(x, y) \ \forall (x, y) \in [\perp, a] \times L$.

vii) $y \leq V(x, y) \ \forall (x, y) \in L \times [\perp, a]$.

viii) $x \vee y \leq V(x, y) \ \forall (x, y) \in [\perp, a]^2$.

ix) $V(x,y) \leq x \wedge y \; \forall (x,y) \in [a,\top]^2$.

x) $(x \wedge a) \vee (y \wedge a) \leq V(x,y) \; \forall (x,y) \in [\bot,a] \times N_a \cup N_a \times [\bot,a] \cup N_a \times N_a$.

xi) $V(x,y) \leq (x \vee a) \wedge (y \vee a) \; \forall (x,y) \in [a,\top] \times N_a \cup N_a \times [a,\top] \cup N_a \times N_a$.

3.4.2 Construction methods

Figure 3-8 show that a nullnorm with zero element a on the unit interval is acting as t-conorm on $[0,a]^2$ and t-norm on $[a,1]^2$ while on the remaining parts of the unit square, nullnorms return as the output the zero element a. It means that, for any t-norm T and any t-conorm S on $[0,1]$ we can obtain a unique nullnorm V (which is false in the uninorm case) with zero element $a \in [0,1]$.

In the case of a bounded lattice $(L, \leq, \bot, \top)$, there may exist one or more elements incomparable with the zero element $a \in L$ and hence the structure of nullnorms on bounded lattices is different from the given ones on the unit interval (see Figure 3-9). There exist many attempts for constructing nullnorms on bounded lattices and we use the following theorems to recall all of these constructions. We start by three constructions given in [48].

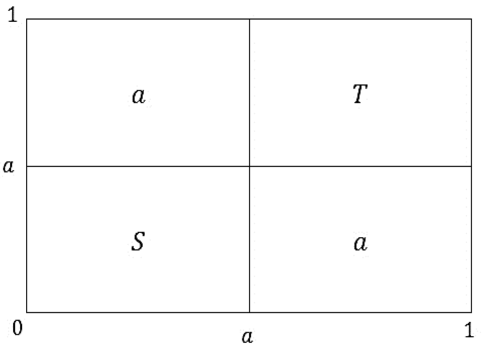

Figure 3-8 The structure of nullnorms on [0,1]

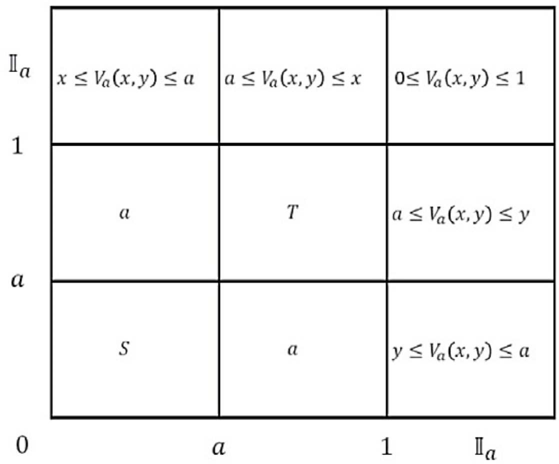

Figure 3-9 The structure of nullnorms on bounded lattices

Theorem 3.8: ([48])

Let $(L, \leq, \perp, \top)$ be a bounded lattice and let $a \in L \backslash \{\perp, \top\}$, S be a t-conorm on $[\perp, a]$, T be a t-norm on $[a, \top]$. Then, the functions $V_S, V_T : L^2 \to L$ defined as follows:

$$V_S(x,y) = \begin{cases} S(x,y) & if\ (x,y) \in [\perp, a]^2, \\ a & if\ (x,y) \in [a, \top[^2 \cup [a, \top] \times N_a \cup N_a \times [a, \top] \cup D_a, \\ S(x \wedge a, y \wedge a) & if\ (x,y) \in [\perp, a] \times N_a \cup N_a \times [\perp, a] \cup N_a \times N_a, \\ x \wedge y & otherwise. \end{cases}$$

(3.15)

$$V_T(x,y)=\begin{cases}T(x,y) & if\ (x,y)\in[a,\top]^2,\\ a & if\ (x,y)\in[\bot,a[^2\cup[\bot,a]\times N_a\cup N_a\times[\bot,a]\cup D_a,\\ T(x\vee a,y\vee a) & if\ (x,y)\in[a,\top]\times N_a\cup N_a\times[a,\top]\cup N_a\times N_a,\\ x\vee y & otherwise.\end{cases}$$

(3.16)

are nullnorms on L with zero element a, where,

$$D_a=[\bot,a[\times]a,\top]\cup]a,\top]\times[\bot,a[,\ N_a=\{x\in L|x\parallel a\}.$$

Proposition 3.5: ([48])

Let $(L,\leq,\bot,\top)$ be a bounded lattice and let $a\in L\backslash\{\bot,\top\}$, S be a t-conorm on $[\bot,a]$, T be a t-norm on $[a,\top]$. Then the function $V_{(T,S)}\colon L^2\to L$ defined as follow:

$$V_{(T,S)}(x,y)=\begin{cases}S(x,y) & if\ (x,y)\in[\bot,a]^2,\\ T(x,y) & if\ (x,y)\in[a,\top]^2,\\ a & otherwise.\end{cases}\qquad(3.17)$$

is a nullnorm on L with zero element a.

Another two constructions for nullnorms on bounded lattices were presented in [35] as follows:

Theorem 3.9: ([35])

Let $(L,\leq,\bot,\top)$ be a bounded lattice, and let $a\in L\backslash\{\bot,\top\}$, S be a t-conorm on $[\bot,a]$, T be a t-norm on $[a,\top]$. Then, the functions $V_T^S,V_S^T\colon L^2\to L$ defined as follows:

$$V_T^S(x,y)=\begin{cases}S(x,y) & if\ (x,y)\in[\bot,a]^2,\\ T(x,y) & if\ (x,y)\in[a,\top]^2,\\ S(x\wedge a,y\wedge a) & if\ (x,y)\in[\bot,a]\times N_a\cup N_a\times[\bot,a]\cup N_a\times N_a,\\ a & otherwise.\end{cases}$$

(3.18)

$$V_S^T(x,y) = \begin{cases} S(x,y) & if\ (x,y) \in [\bot, a]^2, \\ T(x,y) & if\ (x,y) \in [a,\top]^2, \\ T(x \vee a, y \vee a) & if\ (x,y) \in [a,\top] \times N_a \cup N_a \times [a,\top] \cup N_a \times N_a, \\ a & otherwise. \end{cases}$$

(3.19)

are nullnorms on L with zero element a.

Example 3.9:

Consider the bounded lattice $(L, \leq, \bot, \top)$ in Figure 3-10. Let $T^{[a,\top]} = T_D^{[a,\top]}$ and $S^{[\bot,a]} = S_D^{[\bot,a]}$. Then the functions $V_S, V_T, V_{(T,S)}, V_T^S$ and V_S^T whose values are written in Tables 3-14, 3-15, 3-16, 3-17 and 3-18 are nullnorms on L with zero element a. They are constructed using Equations (3.15), (3.16), (3.17), (3.18) and (3.19), respectively.

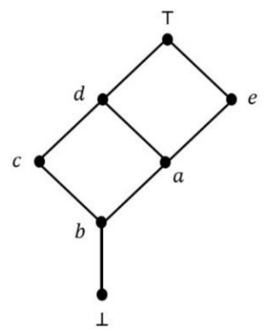

Figure 3-10 The lattice L of Example 3.9

Table 3-14 The nullnorm V_S on L of Example 3.9

V_S	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	a	b	b	a	a	a
a	a	a	a	a	a	a	a
b	b	a	a	a	a	a	a
c	b	a	a	a	a	a	a
d	a	a	a	a	a	a	d
e	a	a	a	a	a	a	e
$\top$	a	a	a	a	d	e	$\top$

Table 3-15 The nullnorm V_T on L of Example 3.9

V_T	$\bot$	a	b	c	d	e	$\top$
$\bot$	a	a	a	a	a	a	a
a	a	a	a	a	a	a	a
b	a	a	a	a	a	a	a
c	a	a	a	a	a	a	d
d	a	a	a	a	a	a	d
e	a	a	a	a	a	a	e
$\top$	a	a	a	d	d	e	$\top$

Table 3-16 The nullnorm $V_{(T,S)}$ on L of Example 3.9

$V_{(T,S)}$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	a	b	a	a	a	a
a	a	a	a	a	a	a	a
b	b	a	a	a	a	a	a
c	a	a	a	a	a	a	a
d	a	a	a	a	a	a	d
e	a	a	a	a	a	a	e
$\top$	a	a	a	a	d	e	$\top$

Table 3-17 The nullnorm V_T^S on L of Example 3.9

V_T^S	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	a	b	b	a	a	a
a	a	a	a	a	a	a	a
b	b	a	a	a	a	a	a
c	b	a	a	a	a	a	a
d	a	a	a	a	a	a	d
e	a	a	a	a	a	a	e
$\top$	a	a	a	a	d	e	$\top$

Table 3-18 The nullnorm V_S^T on L of Example 3.9

V_S^T	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	a	b	a	a	a	a
a	a	a	a	a	a	a	a
b	b	a	a	a	a	a	a
c	a	a	a	a	a	a	a
d	a	a	a	a	a	a	d
e	a	a	a	a	a	a	e
$\top$	a	a	a	a	d	e	$\top$

On the other hand, the construction of idempotent nullnorms on bounded lattices have also attracted much attention from authors [12, 13, 16] . Note that, all construction methods introduced in Theorem 3.8 and Theorem 3.9 for nullnorms on bounded lattices are not suitable for obtaining idempotent nullnorms on bounded lattices. However, $V_{(T,S)}$ introduced in Proposition 3.5 can be used to construct idempotent nullnorms on bounded lattices, if and only if, all elements of L are comparable with the zero element a. Consequently, $V_{(T,S)}$ is reduced to

$$V_{(T,S)}(x,y)=\begin{cases} S(x,y) & if\ (x,y)\in[\bot,a]^2,\\ T(x,y) & if\ (x,y)\in[a,\top]^2,\\ a & if\ (x,y)\in[\bot,a]\times[a,\top]\cup[a,\top]\times[\bot,a]. \end{cases}$$

Hence, if we put $S=S_M^L$ and $T=T_M^L$ in the previous formula, we obtain the following idempotent nullnorm on L:

$$V(x,y)=\begin{cases} x\vee y & if\ (x,y)\in[\bot,a]^2,\\ x\wedge y & if\ (x,y)\in[a,\top]^2,\\ a & if\ (x,y)\in[\bot,a]\times[a,\top]\cup[a,\top]\times[\bot,a]. \end{cases}$$

Moreover, in [13, 16], a characterization of idempotent nullnorms on bounded lattices such that there is only one element in L incomparable with the zero element a has been introduced as follows:

Theorem 3.10: ([13, 16])

Let $(L,\leq,\bot,\top)$ be a bounded lattice and let $a\in L\backslash\{\bot,\top\}$ and suppose there is only one element m in L incomparable with a. Then the following function $V_I\colon L^2\to L$ is an idempotent nullnorm with zero element a.

$$V_I(x,y) = \begin{cases} x \vee y & if\ (x,y) \in [\perp, a]^2, \\ x \wedge y & if\ (x,y) \in [a, \top]^2, \\ a & if\ (x,y) \in [\perp, a] \times [a, \top] \cup [a, \top] \times [\perp, a], \\ x \vee (m \wedge a) & if\ x \in [\perp, a]\ and\ y = m, \\ y \vee (m \wedge a) & if\ x = m\ and\ y \in [\perp, a], \\ x \wedge (m \vee a) & if\ x \in [a, \top]\ and\ y = m, \\ y \wedge (m \vee a) & if\ x = m\ and\ y \in [a, \top], \\ m & if\ x = y = m. \end{cases}$$

(3.20)

Example 3.10:

Consider the bounded lattice $(L, \leq, 0, 1)$ in Figure 3-11. The function V_I in Table 3-19 is an idempotent nullnorm on L with zero element a. It is constructed using Equation (3.20).

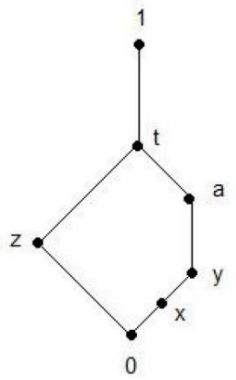

Figure 3-11 The lattice L of Example 3.10

Table 3-19 The idempotent nullnorm V_I on L of Example 3.10

V_I	0	x	y	a	z	t	1
0	0	x	y	a	0	a	a
x	x	x	y	a	x	a	a
y	y	y	y	a	y	a	a
a	a	a	a	a	a	a	a
z	0	x	y	a	z	t	t
t	a	a	a	a	t	t	t
1	a	a	a	a	t	t	1

Remark 3.6:

It is worth mentioning that; the ordinal sum approach has been also introduced for copulas [61, 62] and for general algebraic structures in [6, 8].

As we have seen, the ordinal sum construction method have long been blamed for their limitations in constructing new associative aggregation operator for its inability to cope with a general bounded lattice. Therefore, we aim in the next chapters to present construction methods for t-norms, t-conorms, uninorms and nullnorms on bounded lattices based on the lattice-based sum approach.

CHAPTER FOUR

LATTICE-BASED SUM CONSTRUCTION OF NULLNORMS ON BOUNDED LATTICES

Chapter four
Lattice-based sum construction of nullnorms on bounded lattices

4.1 Introduction

In this chapter, we develop new construction methods for building nullnorms on bounded lattices based on the lattice-based sum of bounded lattices just described in chapter two. Note that, as we have explained in chapter two, we restrict our consideration to the finite lattice-ordered index set where each summand of the associated family is a bounded lattice. In this case, the zero element of the nullnorm may be equal to one of the boundaries of some summand or inside some summand. Therefore, we restrict our consideration about the location of the zero element to be one of the boundaries of some summand. We will illustrate what will happen if the zero element is inside some summand. In addition, we give a new construction method for idempotent nullnorms on bounded lattices with zero element a to be an arbitrary point of the underlying lattice without any restrictions on the zero element a or on the underlying bounded lattice L. In the literature, for the nullnorm V on a bounded lattice L to be idempotent, we need the underlying bounded lattice L to be distributive or there exists only one element on L incomparable with a. By our construction methods obtained in this chapter, we can also obtain t-norms and t-conorms from a given family of t-norms and t-conorms on L, just by controlling the location of the zero element a.

Definition 4.1: ([19])

Let $(L, \leq, \perp, \top)$ be a bounded lattice and $a \in L$. The downset of a denoted $\downarrow a$ and the upset of a denoted $\uparrow a$ are given by $\downarrow a = \{x \in L | x \leq a\}$, $\uparrow a = \{x \in L | x \geq a\}$

4.2 Construction of nullnorms on bounded lattices

Remark 4.1:

Under the consideration of finite lattice-ordered index set where each summand of the associated family is a bounded lattice, we have for some finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and for some $\alpha \in \Lambda$, for any t-norm T_α and any t-conorm S_α on L_α,

$$T_\alpha(x, y) = x \wedge y \text{ and } S_\alpha(x, y) = x \vee y$$

when x or y is equal to one of the boundaries of L_α.

Lemma 4.1: ([31])

Let $(\Lambda, \sqsubseteq)$ be a finite lattice-ordered index set and let $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$ be a lattice-based sum of bounded lattices. Assume that there exist $x_1, x_2 \in L$ such that there is no $\alpha \in \Lambda$ such that $\{x_1, x_2\} \subseteq L_\alpha$

i) If $x_1 < x_2$, then there exist $\alpha_1, \alpha_2 \in \Lambda$ such that $(x_1, x_2) \in L_{\alpha_1} \times L_{\alpha_2}$ with $\alpha_1 \sqsubset \alpha_2$ and for all $z_1 \in L_{\alpha_1}$ and for all $z_2 \in L_{\alpha_2}$ we have $z_1 \leq z_2$.

ii) If $x_1 \parallel x_2$, then for all $\alpha_1 \in I_{x_1}$ and $\alpha_2 \in I_{x_2}$ we have $\alpha_1 \parallel \alpha_2$ and for all $z_1 \in L_{\alpha_1} \backslash \{\perp_{\alpha_1}, \top_{\alpha_1}\}$ and for all $z_2 \in L_{\alpha_2} \backslash \{\perp_{\alpha_2}, \top_{\alpha_2}\}$ we have $z_1 \parallel z_2$.

Lemma 4.1 is a direct consequence from Definition 2.14 and Theorem 2.2.

Example 4.1:

Consider the Λ-sum family of bounded lattices in Figure 2.7. It is clear that, for all $x \in L_\alpha$ and $y \in L_\beta$ we have $x \leq y$ (since $\alpha \sqsubset \beta$). Further, for all $a \in L_\beta \backslash \{\bot_\beta, \top_\beta\}$ and $b \in L_\gamma \backslash \{\bot_\gamma, \top_\gamma\}$ we have $a \parallel b$ (since $\beta \parallel \gamma$).

Theorem 4.1:

Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and let $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \bot_\alpha, \top_\alpha)$ be a lattice-based sum of bounded lattices. Let $a \in L$ with $a \in \{\bot_\alpha, \top_\alpha\}$ for some $\alpha \in \Lambda$ and $(T_\alpha)_{\alpha \in \Lambda}$ ($(S_\alpha)_{\alpha \in \Lambda}$) be a family of t-norms (t-conorms) on the corresponding summands $(L_\alpha)_{\alpha \in \Lambda}$. Then the functions $V_\vee : L^2 \to L$ and $V_\wedge : L^2 \to L$ defined as follow

$$V_\vee(x, y) = \begin{cases} S_\alpha(x, y) & if\ x, y \in L_\alpha \cap \downarrow a, \\ T_\beta(x, y) & if\ x, y \in L_\beta \cap \uparrow a, \\ x \wedge y & if\ x \in L_\alpha \cap \uparrow a, y \in L_\beta \cap \uparrow a, \alpha \neq \beta, \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases} \quad (4.1)$$

and

$$V_\wedge(x, y) = \begin{cases} S_\alpha(x, y) & if\ x, y \in L_\alpha \cap \downarrow a, \\ T_\beta(x, y) & if\ x, y \in L_\beta \cap \uparrow a, \\ x \vee y & if\ x \in L_\alpha \cap \downarrow a, y \in L_\beta \cap \downarrow a, \alpha \neq \beta, \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases} \quad (4.2)$$

are nullnorms on L^2 with zero element a.

Proof:

The proof runs only for the operation $V_\vee$. The operation $V_\wedge$ has a similar proof.

First, we note that, for all $x, y \in L$ with $x, y \in \downarrow a$ and there is no $\alpha \in \Lambda$ such that $\{x, y\} \subseteq L_\alpha$ then we have $V_\vee(x, y) = (x \wedge a) \vee (y \wedge a) = x \vee y$. Also for all $x \in \uparrow a$ and for $y \parallel a$ or $y \in \downarrow a$, we have:

$V_\vee(x,y) = (x \wedge a) \vee (y \wedge a) = a \vee (y \wedge a) = a$. Therefore, by absorption, we will use this abbreviation without mention.

It is necessary to check that the operation $V_\vee$ is well-defined. A problem can only arise if $(x,y) \in L_\alpha \times L_\beta$ with $x \in L_\alpha \cap L_\beta$ for some $\alpha, \beta \in \Lambda$ and we write,

i. $x, y \in \downarrow a$,
   a) $\alpha \sqsubset \beta$. In this case:
   $V_\vee(x,y) = S_\beta(x,y) = x \vee_\beta y = y$ if we consider that $x, y \in L_\beta$, and $V_\vee(x,y) = x \vee y = y$ if we consider that $x \in L_\alpha$ and $y \in L_\beta$. Thus getting the same result in both cases.
   b) $\alpha \parallel \beta$. In this case we have either $x = \top_\alpha = \top_\beta$ and hence, $V_\vee(x,y) = S_\beta(x,y) = x \vee_\beta y = x \vee y = x$, or $x = \perp_\alpha = \perp_\beta$ and hence, $V_\vee(x,y) = S_\beta(x,y) = x \vee_\beta y = x \vee y = y$.

ii. $x, y \in \uparrow a$. This case is dual to case (i) has a dual proof due to the duality between the t-norm and the t-conorm.

   Now, we need to prove that $V_\vee$ is a nullnorm on L with zero element a.

Commutativity: It is easy to see the commutativity of $V_\vee$ due to the commutativity of the t-norm and the t-conorm defined on each summand , $\wedge$ and $\vee$ on L.

Zero element: We prove that a is the zero element of $V_\vee$. The proof is split into all the possible cases for some $x \in L$, as follows:

i. $x \in \downarrow a$,

a) There exists some $\alpha \in \Lambda$ such that $\{x, a\} \subseteq L_\alpha$, then (from Remark 4.1) we have,

$$V_\vee(x, a) = S_\alpha(x, a) = x \vee_\alpha a = a$$

b) There is no $\alpha \in \Lambda$ such that $\{x, a\} \subseteq L_\alpha$, then,

$$V_\vee(x, a) = x \vee a = a$$

ii. $x \in\uparrow a$. This case has a dual proof of case (i) due to the duality between the t-norm and the t-conorm.

iii. $x \parallel a$. Then directly from the definition of $V_\vee$ we have

$$V_\vee(x, a) = a$$

Monotonicity: We prove that if $x \leq y$ in L, then for all $z \in L$, $V_\vee(x, z) \leq V_\vee(y, z)$. The proof is split into all the possible cases, as follows:

Case (1): Let $x, y \in\downarrow a$. Then we have the following subcases

Subcase 1(a): $z \in\downarrow a$,

i. There exist some $\alpha \in \Lambda$ such that $\{x, y\} \subseteq L_\alpha$. If $z \in L_\alpha$, then monotonicity holds trivially due to the monotonicity of S_α on L_α. If $z \notin L_\alpha$, then

$$V_\vee(x, z) = x \vee z \leq y \vee z = V_\vee(y, z)$$

ii. There is no $\alpha \in \Lambda$ such that $\{x, y\} \subseteq L_\alpha$.

a) If x and z are in the same summand, we observe it by considering $\{x, z\} \subseteq L_\beta$ and $y \in L_\delta$ with $\beta \neq \delta$ for some $\beta, \delta \in \Lambda$, then from Lemma 4.1, we have either $\beta \sqsubset \delta$ or $\beta \parallel \delta$. If $\beta \sqsubset \delta$, then

$$V_\vee(x, z) = S_\beta(x, z) \leq y = y \vee z = V_\vee(y, z)$$

If $\beta \parallel \delta$, then we have either $x \in \{\perp_\beta, \top_\beta\}$ and hence,

$$V_\vee(x, z) = S_\beta(x, z) = x \vee z \leq y \vee z = V_\vee(y, z)$$

or $x \in L_\beta \backslash \{\bot_\beta, \top_\beta\}$, then necessarily $y = \top_\delta$ and hence,

$V_\vee(x,z) = S_\beta(x,z) \leq y = y \vee z = V_\vee(y,z)$

b) If y and z are in the same summand, we observe it by considering $\{y,z\} \subseteq L_\alpha$ and $x \notin L_\alpha$ for some $\alpha \in \Lambda$. Therfore

$$V_\vee(x,z) = x \vee z \leq y \vee z \leq S_\alpha(y,z) = V_\vee(y,z)$$

iii. All arguments are in different summands,

$$V_\vee(x,z) = x \vee z \leq y \vee z = V_\vee(y,z)$$

Subcase 1(b): $z \in \uparrow a \Longrightarrow V_\vee(x,z) = a = V_\vee(y,z)$

Subcase 1(c): $z \parallel a$, then

$$V_\vee(x,z) = (x \wedge a) \vee (z \wedge a) \leq (y \wedge a) \vee (z \wedge a) = V_\vee(y,z)$$

Case (2): Let $x \in \uparrow a$. Then $y \in \uparrow a$, and we have:

i. $z \in \downarrow a$, then $V_\vee(x,z) = a = V_\vee(y,z)$

ii. $z \in \uparrow a$. In this case, the proof is a dual proof of Case (1) due to the duality between the t-norm and the t-conorm.

iii. $z \parallel a$, then $V_\vee(x,z) = a = V_\vee(y,z)$

Case (3): Let $x \in \downarrow a, y \in \uparrow a$.

i. $z \in \downarrow a$. In this case we have either x and z are in the same summand or x and z are in different summands. In both cases and due to the t-conorm defined on each summand and $\vee$ on L we have,

$$V_\vee(x,z) \leq a = V_\vee(y,z)$$

ii. $z \in \uparrow a$. Similarly, as in case (i) we have $V_\vee(y,z) \geq a$ and hence,

$$V_\vee(x,z) = a \leq V_\vee(y,z)$$

iii. $z \parallel a$,

$$V_\vee(x,z) = (x \wedge a) \vee (z \wedge a) \leq a = V_\vee(y,z)$$

Case (4): Let $x \in \downarrow a, y \parallel a$.

i. $z \in \downarrow a$,

a) There exists some $\alpha \in \Lambda$ such that $\{x, z\} \subseteq L_\alpha$. Then we have either $y \wedge a \in L_\alpha$ or $y \wedge a \notin L_\alpha$. If $y \wedge a \in L_\alpha$, then necessarily $y \in \{\bot_\alpha, \top_\alpha\}$. In case $y \wedge a = \top_\alpha$, then

$$V_\vee(x, z) = S_\alpha(x, z) \le \top_\alpha = y \wedge a = (y \wedge a) \vee z$$
$$= V_\vee(y, z)$$

In case $y \wedge a = \bot_\alpha$, then necessarily $x = \bot_\alpha$ and hence

$$V_\vee(x, z) = S_\alpha(x, z) = x \vee z = z = (y \wedge a) \vee z$$
$$= V_\vee(y, z)$$

If $y \wedge a \notin L_\alpha$, then $y \wedge a > u$ for all $u \in L_\alpha$ and hence,

$$V_\vee(x, z) = S_\alpha(x, z) \le y \wedge a = (y \wedge a) \vee z = V_\vee(y, z)$$

b) There is no $\alpha \in \Lambda$ such that $\{x, z\} \subseteq L_\alpha$, then

$$V_\vee(x, z) = (x \wedge a) \vee (z \wedge a) \le (y \wedge a) \vee (z \wedge a)$$
$$= V_\vee(y, z)$$

ii. $z \in \uparrow a$, then $V_\vee(x, z) = a = V_\vee(y, z)$

iii. $z \parallel a$. This case is similar to subcase 1(c) resulting in a similar proof.

Case (5): Let $x \parallel a, y \in \uparrow a$.

i. $z \in \downarrow a$,

$$V_\vee(x, z) = (x \wedge a) \vee (z \wedge a) \le a = V_\vee(y, z)$$

ii. $z \in \uparrow a$. In similar way of Case 3(ii) we have $V_\vee(y, z) \ge a$ and hence,

$$V_\vee(x, z) = a \le V_\vee(y, z)$$

iii. $z \parallel a$,

$$V_\vee(x, z) = (x \wedge a) \vee (z \wedge a) \le a = V_\vee(y, z)$$

Case (6): Let $x \parallel a, y \parallel a$.

i. $z \in \downarrow a$. This case is similar to Case 4(iii) has a similar proof.
ii. $z \in \uparrow a$,then $V_{\vee}(x,z) = a = V_{\vee}(y,z)$
iii. $z \parallel a$,

$$V_{\vee}(x,z) = (x \wedge a) \vee (z \wedge a) \leq (y \wedge a) \vee (z \wedge a) = V_{\vee}(y,z)$$

Associativity: We prove that $V_{\vee}\,(V_{\vee}(x,y),z) = V_{\vee}\big(x, V_{\vee}(y,z)\big)$ for all $x, y, z \in L$. Again, the proof is split into all possible cases by considering the relationship between the arguments x, y, z and a, as follows:

Case (1): All arguments are from $\downarrow a$.

i. There exists some $\alpha \in \Lambda$ such that $\{x, y, z\} \subseteq L_{\alpha}$. In this case associativity holds trivially due to the associativity of S_{α} on L_{α}.
ii. All arguments are from different summands,

$$\begin{aligned} V_{\vee}\,(V_{\vee}(x,y),z) &= V_{\vee}\,(x \vee y, z) = x \vee y \vee z \\ &= V_{\vee}\,(x, y \vee z) = V_{\vee}\big(x, V_{\vee}(y,z)\big) \end{aligned}$$

In this case, we must note that, if $x \vee y$ and z are in the same summand, then necessarily $x \vee y$ is equal to one of the boundaries of this summand and hence (from Remark 4.1) we have $V_{\vee}\,(x \vee y, z) = x \vee y \vee z$.
iii. Exactly two arguments are from the same summand. We observe it by considering the following cases.
   a) There exists some $\alpha \in \Lambda$ such that $\{x, y\} \subseteq L_{\alpha}$ and $z \notin L_{\alpha}$. If x or y is equal to one of the boundaries of L_{α}, then (from Remark 4.1) associativity holds trivially due to the associativity of $\vee$ on L. Therefore, we assume that $x, y \in L_{\alpha} \backslash \{\perp_{\alpha}, \top_{\alpha}\}$ and then we compare z with x and y, as follows:

If $x > z$ or $y > z$, then

$$\begin{aligned} V_{\vee}\ (V_{\vee}(x,y),z) &= V_{\vee}(S_{\alpha}(x,y),z) = S_{\alpha}(x,y) \vee z \\ &= S_{\alpha}(x,y) = S_{\alpha}(x,y \vee z) \\ &= V_{\vee}\big(x, V_{\vee}(y,z)\big) \end{aligned}$$

If $x < z$ or $y < z$, then

$$\begin{aligned} V_{\vee}\ (V_{\vee}(x,y),z) &= V_{\vee}(S_{\alpha}(x,y),z) = S_{\alpha}(x,y) \vee z \\ &= z = y \vee z = V_{\vee}(y,z) \\ &= x \vee V_{\vee}(y,z) = V_{\vee}\big(x, V_{\vee}(y,z)\big) \end{aligned}$$

If $x \parallel z$ or $y \parallel z$, then $x \vee z = y \vee z = S_{\alpha}(x,y) \vee z$ and hence,

$$\begin{aligned} V_{\vee}\ (V_{\vee}(x,y),z) &= V_{\vee}(S_{\alpha}(x,y),z) = S_{\alpha}(x,y) \vee z \\ &= y \vee z = V_{\vee}(y,z) \\ &= x \vee V_{\vee}(y,z) = V_{\vee}\big(x, V_{\vee}(y,z)\big) \end{aligned}$$

b) There exist some $\beta \in \Lambda$ such that $\{x,z\} \subseteq L_{\beta}$ and $y \notin L_{\beta}$. This case is similar to Case (a) resulting in similar proof.

c) There exist some $\delta \in \Lambda$ such that $\{y,z\} \subseteq L_{\delta}$ and $x \notin L_{\delta}$. This case is similar to Case (a) resulting similar proof.

Case (2): All arguments are from $\uparrow a$. This case has a dual proof of Case (1) due to the duality between t-norm and t-conorm.

Case (3): All arguments are incomparable with a,

$$\begin{aligned} V_{\vee}\ (V_{\vee}(x,y),z) &= V_{\vee}\Big(\big((x \wedge a) \vee (y \wedge a)\big), z\Big) \\ &= (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) \\ &= V_{\vee}\big(x, (y \wedge a) \vee (z \wedge a)\big) = V_{\vee}\big(x, V_{\vee}(y,z)\big) \end{aligned}$$

Case (4): Exactly two arguments are from $\downarrow a$.

i. $x, y \in \downarrow a, z > a$. In this case we have either x and y are in the same summand or x and y are from different summands. In both cases, we have $V_\vee(x, y) \leq a$ and hence,

$$V_\vee\,(V_\vee(x, y), z) = a = V_\vee(x, a) = V_\vee\big(x, V_\vee(y, z)\big)$$

ii. $x, y \in \downarrow a, z \parallel a$. Then from the fact that $z \wedge a < a$, the associativity holds by a proof exactly similar to Case (1) but with $x, y \in \downarrow a$ and $z \wedge a < a$.

iii. $x, z \in \downarrow a, y > a$,

$$V_\vee\,(V_\vee(x, y), z) = V_\vee(a, z) = a = V_\vee(x, a) = V_\vee\big(x, V_\vee(y, z)\big)$$

iv. $x, z \in \downarrow a, y \parallel a$. This case is similar to Case 4(ii) resulting in similar proof.

v. $y, z \in \downarrow a, x > a$. This case is similar to Case 4(i) resulting in similar proof.

vi. $y, z \in \downarrow a, x \parallel a$. This case is similar to Case 4(ii) resulting in similar proof.

Case (5): Exactly two arguments are from $\uparrow a$.

i. $x, y \in \uparrow a, z < a$. In this case we have either x and y are in the same summand or x and y are in different summands. In both cases, we have $V_\vee(x, y) \geq a$ and hence

$$V_\vee\,(V_\vee(x, y), z) = a = V_\vee(x, a) = V_\vee\big(x, V_\vee(y, z)\big)$$

ii. $x, y \in \uparrow a, z \parallel a$. This case is similar to Case 5(i) resulting in similar proof.

iii. $x, z \in \uparrow a, y < a$.

$$V_\vee\,(V_\vee(x, y), z) = V_\vee(a, z) = a = V_\vee(x, a) = V_\vee\big(x, V_\vee(y, z)\big)$$

iv. $x, z \in \uparrow a, y \parallel a$. This case is similar to Case 5(iii) resulting in similar proof.

v. $y, z \in \uparrow a, x < a$. In this case we have either y and z are in the same summand or y and z are in different summands. In both cases, we have $V_{\vee}(y, z) \geq a$ and hence

$$V_{\vee}\,(V_{\vee}(x, y), z) = V_{\vee}(x, a) = a = V_{\vee}\big(x, V_{\vee}(y, z)\big)$$

vi. $y, z \in \uparrow a, x \parallel a$. This case is similar to Case 5(v) resulting in similar proof.

Case (6): Exactly two arguments are incomparable with a.

i. $x \parallel a, y \parallel a, z \in \downarrow a$.

In this case we have $x \wedge a < a$ and $y \wedge a < a$ with $x \wedge a$ and $y \wedge a$ are on the boundaries and hence (from Remark 4.1) we have

$$\begin{aligned} V_{\vee}\,(V_{\vee}(x, y), z) &= V_{\vee}\Big(\big((x \wedge a) \vee (y \wedge a)\big), z\Big) \\ &= (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) \\ &= V_{\vee}\big(x, (y \wedge a) \vee (z \wedge a)\big) \\ &= V_{\vee}\big(x, V_{\vee}(y, z)\big) \end{aligned}$$

ii. $x \parallel a, y \parallel a, z \in \uparrow a$.

$$\begin{aligned} V_{\vee}\,(V_{\vee}(x, y), z) = V_{\vee}\Big(\big((x \wedge a) \vee (y \wedge a)\big), z\Big) = a = V_{\vee}(x, a) \\ = V_{\vee}\big(x, V_{\vee}(y, z)\big) \end{aligned}$$

iii. $x \parallel a, z \parallel a, y \in \downarrow a$. This case is similar to Case 6(i) resulting in similar proof.

iv. $x \parallel a, z \parallel a, y \in \uparrow a$,

$$V_{\vee}\,(V_{\vee}(x, y), z) = V_{\vee}(a, z) = a = V_{\vee}(x, a) = V_{\vee}\big(x, V_{\vee}(y, z)\big)$$

v. $y \parallel a, z \parallel a, x \in \downarrow a$. This case is similar to Case 6(i) resulting in similar proof.

vi. $y \parallel a, z \parallel a, x \in \uparrow a$. This case is similar to Case 6(iv) resulting in similar proof.

For other possibilities we distinguish the following cases

i. $x \in \downarrow a, y \in \uparrow a, z \parallel a,$

$$V_\vee\ (V_\vee(x,y),z) = V_\vee(a,z) = a = V_\vee(x,a) = V_\vee\big(x, V_\vee(y,z)\big)$$

ii. $x \in \downarrow a, y \parallel a, z \in \uparrow a,$

$$V_\vee\ (V_\vee(x,y),z) = V_\vee\Big(\big((x \wedge a) \vee (y \wedge a)\big), z\Big) = a = V_\vee(x,a) = V_\vee\big(x, V_\vee(y,z)\big)$$

iii. $x \in \uparrow a, y \in \downarrow a, z \parallel a,$

$$V_\vee\ (V_\vee(x,y),z) = V_\vee(a,z) = a = V_\vee\big(x, (y \wedge a) \vee (z \wedge a)\big) = V_\vee\big(x, V_\vee(y,z)\big)$$

iv. $x \in \uparrow a, y \parallel a, z \downarrow a$. This case is similar to Case (iii) resulting in similar proof.

v. $x \parallel a, y \downarrow a, z \in \uparrow a$. This case is similar to Case (ii) resulting in similar proof.

vi. $x \parallel a, y \uparrow a, z \downarrow a$. This case is similar to Case (i) resulting in similar proof.

Example 4.2:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ and its lattice-based sum of bounded lattices L of Example 2.3 with elements distribution shown in Figure 4-1. Define a drastic product t-norm T_D on L_γ and a drastic sum t-conorm S_D on L_α. Then the functions $V_\vee$ and $V_\wedge$ whose values are written in Table 4-1 and Table 4-2 are nullnorms on L with zero element a. They are constructed using Formulas (4.1) and (4.2), respectively.

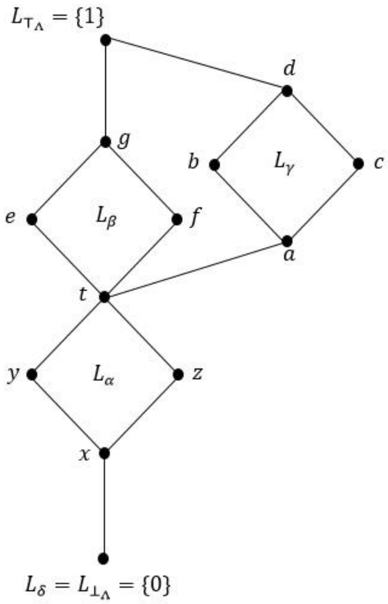

Figure 4-1 The lattice L of Example 4.2

Table 4-1 The nullnorm $V_\vee$ on L of Example 4.2

$V_\vee$	0	x	y	z	t	e	f	g	a	b	c	d	1
0	0	x	y	z	t	t	t	t	a	a	a	a	a
x	x	x	y	z	t	t	t	t	a	a	a	a	a
y	y	y	t	t	t	t	t	t	a	a	a	a	a
z	z	z	t	t	t	t	t	t	a	a	a	a	a
t	t	t	t	t	t	t	t	t	a	a	a	a	a
e	t	t	t	t	t	t	t	t	a	a	a	a	a
f	t	t	t	t	t	t	t	t	a	a	a	a	a
g	t	t	t	t	t	t	t	t	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	a	a	a	a	a	a	a	a	a	a	a	b	b
c	a	a	a	a	a	a	a	a	a	a	a	c	c
d	a	a	a	a	a	a	a	a	a	b	c	d	d
1	a	a	a	a	a	a	a	a	a	b	c	d	1

Table 4-2 The nullnorm $V_\wedge$ on L of Example 4.2

$V_\wedge$	0	x	y	z	t	e	f	g	a	b	c	d	1
0	0	x	y	z	t	a	a	a	a	a	a	a	a
x	x	x	y	z	t	a	a	a	a	a	a	a	a
y	y	y	t	t	t	a	a	a	a	a	a	a	a
z	z	z	t	t	t	a	a	a	a	a	a	a	a
t	t	t	t	t	t	a	a	a	a	a	a	a	a
e	a	a	a	a	a	1	1	1	a	b	c	d	1
f	a	a	a	a	a	1	1	1	a	b	c	d	1
g	a	a	a	a	a	1	1	1	a	b	c	d	1
a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	a	a	a	a	a	b	b	b	a	a	a	b	b
c	a	a	a	a	a	c	c	c	a	a	a	c	c
d	a	a	a	a	a	d	d	d	a	b	c	d	d
1	a	a	a	a	a	1	1	1	a	b	c	d	1

Corollary 4.1:

Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha\in\Lambda} (L_\alpha, \leq_\alpha, \bot_\alpha, \top_\alpha)$. If we put $S_\alpha = S_D^L$ and $T_\alpha = T_D^L$ for all $\alpha \in \Lambda$ in $V_\vee$ and $V_\wedge$ in Theorem 4.1, then we obtain the following nullnorms on L with zero element $a \in L$

$$V_\vee^D(x,y) = \begin{cases} \top_\alpha & if\ x,y \in (L_\alpha \cap\downarrow a)\backslash\{\bot_\alpha\}, \\ \bot_\beta & if\ x,y \in \left(L_\beta \cap\uparrow a\right)\backslash\{\top_\beta\}, \\ x \wedge y & if\ x \in L_\alpha \cap\uparrow a, y \in L_\beta \cap\uparrow a, \alpha \neq \beta, \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases}$$

$$V_\wedge^D(x,y) = \begin{cases} \top_\alpha & if\ x,y \in (L_\alpha \cap\downarrow a)\backslash\{\bot_\alpha\}, \\ \bot_\beta & if\ x,y \in \left(L_\beta \cap\uparrow a\right)\backslash\{\top_\beta\}, \\ x \vee y & if\ x \in L_\alpha \cap\uparrow a, y \in L_\beta \cap\uparrow a, \alpha \neq \beta, \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases}$$

4.3 Construction of idempotent nullnorms on bounded lattices

Remark 4.2:

Given a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \bot_\alpha, \top_\alpha)$ and $a \in L \backslash \{\bot, \top\}$. The nullnorms $V_\vee$ and $V_\wedge$ given in Theorem 4.1 may not work to construct idempotent nullnorms on L for any zero element $a \in L$, such that, for some $x \in L$ with $x \parallel a$ we have

$$V_\vee(x, x) = x \wedge a \neq x \text{ and } V_\wedge(x, x) = x \vee a \neq x$$

Inspired by the last observation, we introduce the following theorem

Theorem 4.2:

Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and let $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \bot_\alpha, \top_\alpha)$ be a lattice-based sum of bounded lattices. Let $a \in L$ with $a \in \{\bot_\alpha, \top_\alpha\}$ for some $\alpha \in \Lambda$ and $(T_\alpha)_{\alpha \in \Lambda}$ $((S_\alpha)_{\alpha \in \Lambda})$ be a family of t-norms (t-conorms) on the corresponding summands $(L_\alpha)_{\alpha \in \Lambda}$. Then the functions $V_\vee^I : L^2 \to L$ and $V_\wedge^I : L^2 \to L$ defined as follow

$$V_\vee^I(x, y) = \begin{cases} S_\alpha(x, y) & if\ x, y \in L_\alpha \cap\downarrow a, \\ T_\beta(x, y) & if\ x, y \in L_\beta \cap\uparrow a, \\ x \wedge y & if\ \left(x \in L_\alpha \cap\uparrow a, y \in L_\beta \cap\uparrow a, \alpha \neq \beta\right)\ or\ (x = y \parallel a), \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases} \tag{4.3}$$

$$V_\wedge^I(x, y) = \begin{cases} S_\alpha(x, y) & if\ x, y \in L_\alpha \cap\downarrow a, \\ T_\beta(x, y) & if\ x, y \in L_\beta \cap\uparrow a, \\ x \vee y & if\ \left(x \in L_\alpha \cap\downarrow a, y \in L_\beta \cap\downarrow a, \alpha \neq \beta\right)\ or\ (x = y \parallel a), \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases} \tag{4.4}$$

are nullnorms on L with zero element a.

Proof:

The proof runs only for the operation $V_\vee^I$. The operation $V_\wedge^I$ has a similar proof.

The commutativity, the monotonicity and the fact that a is the zero element of $V_\vee^I$ have exactly the same proof as the corresponding one from Theorem 4.1. It is only remaining to see the associativity of $V_\vee^I$.

Associativity: We prove that:

$$V_\vee^I\left(V_\vee^I(x,y),z\right)=V_\vee^I\left(x,V_\vee^I(y,z)\right) \text{ for all } x,y,z\in L.$$

Associativity of $V_\vee^I$ is preserved in all cases by exactly the same proof of the corresponding cases from Theorem 4.1, but we investigate if at least two equal arguments are incomparable with a. Therefore, we assume that $x=y \parallel a$ and distinguish the following cases

Case (1): $z \parallel a$ with $z\neq x$ (equivalent to $z\neq y$)

$$\begin{aligned} V_\vee^I\left(V_\vee^I(x,y),z\right) &= V_\vee^I(x\wedge y,z)=V_\vee^I(x,z)=(x\wedge a)\vee(z\wedge a)\\ &=(x\wedge a)\vee(y\wedge a)\vee(z\wedge a)\\ &=V_\vee^I\big(x,(y\wedge a)\vee(z\wedge a)\big)=V_\vee^I\big(x,V_\vee^I(y,z)\big)\end{aligned}$$

Case (2): $z\in\downarrow a$,

$$\begin{aligned} V_\vee^I\left(V_\vee^I(x,y),z\right) &= V_\vee^I(x\wedge y,z)=V_\vee^I(x,z)=(x\wedge a)\vee(z\wedge a)\\ &=(x\wedge a)\vee(y\wedge a)\vee(z\wedge a)\\ &=V_\vee^I\big(x,(y\wedge a)\vee(z\wedge a)\big)=V_\vee^I\big(x,V_\vee^I(y,z)\big)\end{aligned}$$

Case (3): $z\in\uparrow a$,

$$\begin{aligned} V_\vee^I\left(V_\vee^I(x,y),z\right) &= V_\vee^I(x\wedge y,z)=V_\vee^I(x,z)=(x\wedge a)\vee(z\wedge a)\\ &=a=V_\vee^I(x,a)=V_\vee^I(x,(y\wedge a)\vee a)\\ &=V_\vee^I\big(x,(y\wedge a)\vee(z\wedge a)\big)=V_\vee^I\big(x,V_\vee^I(y,z)\big)\end{aligned}$$

All other cases can be shown in a similar way.

Corollary 4.2:

If we put $T_\alpha = T_M^L$ and $S_\alpha = S_M^L$ on L_α for all $\alpha \in \Lambda$ in $V_\vee^I$ and $V_\wedge^I$ in Theorem 4.2, then the following functions are idempotent nullnorms on L.

$$V_\vee^I\,(x,y) = \begin{cases} x \wedge y & if\ (x, y \in\uparrow a)\ or\ (x = y \parallel a), \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases}$$

$$V_\wedge^I\,(x,y) = \begin{cases} x \vee y & if\ (x, y \in\downarrow a)\ or\ (x = y \parallel a), \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases}$$

Note that $V_\vee^I$ and $V_\wedge^I$ just described in Corollary 4.2 are to be considered as a new constructions for idempotent nullnorms on bounded lattices without any restrictions on the zero element a or on the underlying bounded lattice.

Recall that, the nullnorm $V_{(T,S)}$ in Proposition 3.5 is an idempotent nullnorm on a bounded lattice L if and only if the underling bounded lattice L is a chain (i.e. all elements in L are comparable with the zero element a). On the other hand, the nullnorm V_I in Theorem 3.10 is idempotent nullnorm on a bounded lattice L if and only if there is only one element in L incomparable with a.

Example 4.3:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ and its lattice-based sum of bounded lattices L of Example 4.2. Let $T_\alpha = T_M^L$ and $S_\alpha = S_M^L$ on L_α for all $\alpha \in \Lambda$. Then the functions $V_\vee^I$ and $V_\wedge^I$ whose values are written in Table 4-3 and Table 4-4 are idempotent nullnorms on L with zero element a. They are constructed using Formulas (4.3) and (4.4), respectively.

Table 4-3 The idempotent nullnorms $V_{\vee}^{I}$ on L of Example 4.3

$V_{\vee}^{I}$	0	x	y	z	t	e	f	g	a	b	c	d	1
0	0	x	y	z	t	t	t	t	a	a	a	a	a
x	x	x	y	z	t	t	t	t	a	a	a	a	a
y	y	y	y	t	t	t	t	t	a	a	a	a	a
z	z	z	t	z	t	t	t	t	a	a	a	a	a
t	t	t	t	t	t	t	t	t	a	a	a	a	a
e	t	t	t	t	t	e	t	t	a	a	a	a	a
f	t	t	t	t	t	t	f	t	a	a	a	a	a
g	t	t	t	t	t	t	t	g	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	a	a	a	a	a	a	a	a	a	b	a	b	b
c	a	a	a	a	a	a	a	a	a	a	c	c	c
d	a	a	a	a	a	a	a	a	a	b	c	d	d
1	a	a	a	a	a	a	a	a	a	b	c	d	1

Table 4-4 The idempotent nullnorm $V_{\wedge}^{I}$ on L of Example 4.3

$V_{\wedge}^{I}$	0	x	y	z	t	e	f	g	a	b	c	d	1
0	0	x	y	z	t	a	a	a	a	a	a	a	a
x	x	x	y	z	t	a	a	a	a	a	a	a	a
y	y	y	y	t	t	a	a	a	a	a	a	a	a
z	z	z	t	z	t	a	a	a	a	a	a	a	a
t	t	t	t	t	t	a	a	a	a	a	a	a	a
e	a	a	a	a	a	e	1	1	a	b	c	d	1
f	a	a	a	a	a	1	f	1	a	b	c	d	1
g	a	a	a	a	a	1	1	g	a	b	c	d	1
a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	a	a	a	a	a	b	b	b	a	b	a	b	b
c	a	a	a	a	a	c	c	c	a	a	c	c	c
d	a	a	a	a	a	d	d	d	a	b	c	d	d
1	a	a	a	a	a	1	1	1	a	b	c	d	1

Note that, in Example 4.3, although L isn't distributive (since $e \wedge a = f \wedge a$ and $e \vee a = f \vee a$ but $e \neq f$), the obtained nullnorms $V_{\vee}^{I}$ and $V_{\wedge}^{I}$ are idempotent nullnorms on L with the indicated zero element a.

Corollary 4.3:

Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$. If we put $S_\alpha = S_D^L$ and $T_\alpha = T_D^L$ for all $\alpha \in \Lambda$ in $V_{\vee}^{I}$ and $V_{\wedge}^{I}$ in Theorem 4.2, then we obtain the following nullnorms

$$V_{\vee}^{d}(x,y) = \begin{cases} \top_\alpha & if\ x,y \in (L_\alpha \cap\downarrow a)\backslash\{\perp_\alpha\}, \\ \perp_\beta & if\ x,y \in \left(L_\beta \cap\uparrow a\right)\backslash\{\top_\beta\}, \\ x \wedge y & if\ \left(x \in L_\alpha \cap\uparrow a, y \in L_\beta \cap\uparrow a, \alpha \neq \beta\right)\ or\ (x = y \parallel a), \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases}$$

$$V_{\wedge}^{d}(x,y) = \begin{cases} \top_\alpha & if\ x,y \in (L_\alpha \cap\downarrow a)\backslash\{\perp_\alpha\}, \\ \perp_\beta & if\ x,y \in \left(L_\beta \cap\uparrow a\right)\backslash\{\top_\beta\}, \\ x \vee y & if\ \left(x \in L_\alpha \cap\downarrow a, y \in L_\beta \cap\downarrow a, \alpha \neq \beta\right)\ or\ (x = y \parallel a), \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases}$$

Corollary 4.4:

Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$. If we put $S_\alpha = S_M^L$ and $T_\alpha = T_M^L$ for all $\alpha \in \Lambda$ in $V_{\vee}$ and $V_{\wedge}$ in Theorem 4.1, then we obtain the following nullnorms

$$V_{\vee}^{M}(x,y) = \begin{cases} x \wedge y & if x,y \in\uparrow a\,, \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases}$$

and

$$V_{\wedge}^{M}(x,y) = \begin{cases} x \vee y & if\ x,y \in\downarrow a, \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases}$$

Remark 4.3:

The zero element a of the nullnorms $V_{\vee}, V_{\wedge}, V_{\vee}^{I}$ and $V_{\wedge}^{I}$ was restricted to be one of the boundaries of some summand. If a is inside some summand, then $V_{\vee}, V_{\wedge}, V_{\vee}^{I}$ and $V_{\wedge}^{I}$ may not work to construct nullnorms on L. For example, if we consider a lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha \in \Lambda} (L_{\alpha}, \leq_{\alpha}, \perp_{\alpha}, \top_{\alpha})$ and there exists some $\alpha \in \Lambda$ such that $\{x, y, a\} \subseteq L_{\alpha}$ with $\perp_{\alpha} < x < a < y < \top_{\alpha}$ and $T_{\alpha} = T_{D}^{L}, S_{\alpha} = S_{D}^{L}$, then from Theorem 4.1 and Theorem 4.2 we have:

$$\begin{aligned} V_{\vee}(x, a) &= V_{\wedge}(x, a) = V_{\vee}^{I}(x, a) = V_{\wedge}^{I}(x, a) \\ &= S_{\alpha}(x, a) = S_{D}^{L}(x, a) = \top_{\alpha} \neq a \end{aligned}$$

$$\begin{aligned} V_{\vee}(y, a) &= V_{\wedge}(y, a) = V_{\vee}^{I}(y, a) = V_{\wedge}^{I}(y, a) \\ &= T_{\alpha}(y, a) = T_{D}^{L}(y, a) = \perp_{\alpha} \neq a \end{aligned}$$

This violates the zero element property of the nullnorm operator. However, the functions $V_{\vee}, V_{\wedge}, V_{\vee}^{I}$ and $V_{\wedge}^{I}$ are still nullnorms on L in case a is inside some summand if and only if the t-norm and the t-conorm defined on this summand are fixed to be the minimum T_{M}^{L} and the maximum S_{M}^{L}, respectively.

4.4 More illustrative examples

Example 4.4:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ shown in Figure 4-2 and the lattice-based sum of bounded lattices L shown in Figure 4-3 where $L_{\perp_{\Lambda}} = \{0\}$, $L_{\alpha} = \{x, y, z, t, g\}$, $L_{\beta} = \{g\}$, $L_{\delta} = \{a, b, c, d, e, f\}$, and $L_{\top_{\Lambda}} = \{g, h, m, n, 1\}$. Let $T_{\top_{\Lambda}}$ be the t-norm defined on $L_{\top_{\Lambda}}$ whose values are written in Table 4-5 and S_{δ} be the t-conorm on L_{δ} whose values are written in Table 4-6. Then the functions $V_{\vee}$ and $V_{\wedge}$ which values are written in Table 4-7 and Table 4-8, respectively, are nullnorms on L with zero element a.

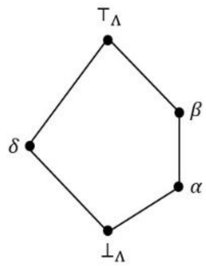

Figure 4-2 The lattice $(\Lambda, \sqsubseteq)$ of Example 4.4

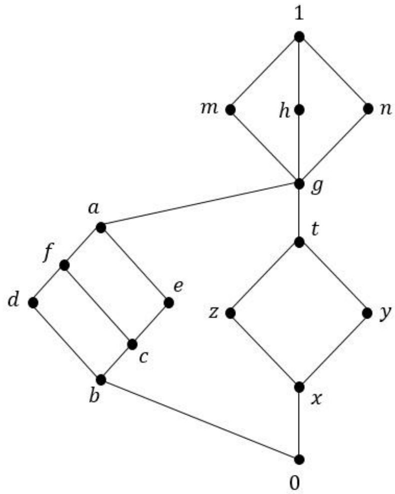

Figure 4-3 The lattice L of Example 4.4

Table 4-5 The t-norm $T_{\top_\Lambda}$ on $L_{\top_\Lambda}$

$T_{\top_\Lambda}$	g	h	m	n	1
g	g	g	g	g	g
h	g	h	g	g	h
m	g	g	g	g	m
n	g	g	g	g	n
1	g	h	m	n	1

Table 4-6 The t-conorm S_δ on L_δ

S_δ	b	c	d	e	f	a
b	b	c	d	e	f	a
c	c	f	f	a	f	a
d	d	f	f	a	f	a
e	e	a	a	a	a	a
f	f	f	f	a	f	a
a	a	a	a	a	a	a

Table 4-7 The nullnorm $V_{\vee}$ on L of Example 4.4

$V_{\vee}$	0	x	y	z	t	g	b	c	d	e	f	a	h	m	n	1
0	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
x	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
y	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
z	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
t	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
g	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	g
b	b	b	b	b	b	a	b	c	d	e	f	a	a	a	a	a
c	c	c	c	c	c	a	c	f	f	a	f	a	a	a	a	a
d	d	d	d	d	d	a	d	f	f	a	f	a	a	a	a	a
e	e	e	e	e	e	a	e	a	a	a	a	a	a	a	a	a
f	f	f	f	f	f	a	f	f	f	a	f	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
h	a	a	a	a	a	g	a	a	a	a	a	a	h	g	g	h
m	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	m
n	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	n
1	a	a	a	a	a	g	a	a	a	a	a	a	h	m	n	1

Table 4-8 The nullnorm $V_{\wedge}$ on L of Example 4.4

$V_{\wedge}$	0	x	y	z	t	g	b	c	d	e	f	a	h	m	n	1
0	0	a	a	a	a	a	b	c	d	e	f	a	a	a	a	a
x	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
y	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
z	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
t	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
g	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
b	b	a	a	a	a	a	b	c	d	e	f	a	a	a	a	a
c	c	a	a	a	a	a	c	f	f	a	f	a	a	a	a	a
d	d	a	a	a	a	a	d	f	f	a	f	a	a	a	a	a
e	e	a	a	a	a	a	e	a	a	a	a	a	a	a	a	a
f	f	a	a	a	a	a	f	f	f	a	f	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
h	a	g	g	g	g	g	a	a	a	a	a	a	h	g	g	h
m	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	m
n	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	n
1	a	g	g	g	g	g	a	a	a	a	a	a	h	m	n	1

Example 4.5:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ of Example 4.4 and it lattice-based sum of bounded lattices L in Figure 4-4. Let $S_{\perp_\Lambda} = S_D^L$. Then the functions $V_\vee$ and $V_\wedge$ whose values are written in Table 4-9 and Table 4-10, respectively, are nullnorms on L with zero element a. Note that a is inside L_δ, then according to Remark 4.3, the t-norm T_δ and the t-conorm S_δ are considered to be the minimum T_M^L and the maximum S_M^L, respectively.

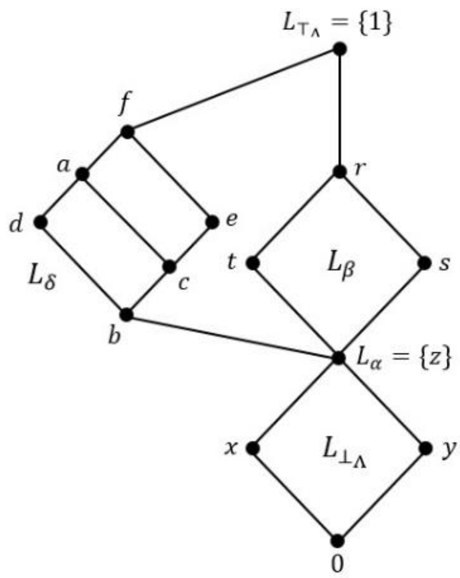

Figure 4-4 The lattice L of Example 4.5

Table 4-9 The nullnorm $V_\vee$ on L of Example 4.5

$V_\vee$	0	x	y	z	t	s	r	a	b	c	d	e	f	1
0	0	x	y	z	z	z	z	a	b	c	d	c	a	a
x	x	z	z	z	z	z	z	a	b	c	d	c	a	a
y	y	z	z	z	z	z	z	a	b	c	d	c	a	a
z	z	z	z	z	z	z	z	a	b	c	d	c	a	a
t	z	z	z	z	z	z	z	a	b	c	d	c	a	a
s	z	z	z	z	z	z	z	a	b	c	d	c	a	a
r	z	z	z	z	z	z	z	a	b	c	d	c	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b	a	b	c	d	c	a	a
c	c	c	c	c	c	c	c	a	c	c	a	c	a	a
d	d	d	d	d	d	d	d	a	d	a	d	a	a	a
e	c	c	c	c	c	c	c	a	c	c	a	c	a	a
f	a	a	a	a	a	a	a	a	a	a	a	a	f	f
1	a	a	a	a	a	a	a	a	a	a	a	a	f	1

Table 4-10 The nullnorm $V_\wedge$ on L of Example 4.5

$V_\wedge$	0	x	y	z	t	s	r	a	b	c	d	e	f	1
0	0	x	y	z	a	a	a	a	b	c	d	a	a	a
x	x	z	z	z	a	a	a	a	b	c	d	a	a	a
y	y	z	z	z	a	a	a	a	b	c	d	a	a	a
z	z	z	z	z	a	a	a	a	b	c	d	a	a	a
t	a	a	a	a	1	1	1	a	a	a	a	f	f	1
s	a	a	a	a	1	1	1	a	a	a	a	f	f	1
r	a	a	a	a	1	1	1	a	a	a	a	f	f	1
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	b	b	b	b	a	a	a	a	b	c	d	a	a	a
c	c	c	c	c	a	a	a	a	c	c	a	a	a	a
d	d	d	d	d	a	a	a	a	d	a	d	a	a	a
e	a	a	a	a	f	f	f	a	a	a	a	f	f	f
f	a	a	a	a	f	f	f	a	a	a	a	f	f	f
1	a	a	a	a	1	1	1	a	a	a	a	f	f	1

4.5 Lattice-based sum construction of t-norms and t-conorms on bounded lattices

The obtained nullnorms in Theorem 4.1 and Theorem 4.2 can be used to construct t-norms and t-conorms from a given family of t-norms and t-conorms on bounded lattices, such that if $a = \top$ we obtain t-conorms and if $a = \perp$ we obtain t-norms. Consequently, we get, as a corollary, the following lattice-based sum constructions of t-norms and t-conorms obtained by El-Zekey [31] in a more general setting where the lattice-ordered index set need not be finite and the so-called t-subnorms (t-subconorms) can be used (with a little restriction) instead of t-norms (t-conorms) as summands in the lattice-based sum construction of t-norms (t-conorms).

Corollary 4.5:

With all the assumptions of Theorem 4.1 and Theorem 4.2 the nullnorm functions $V_\vee, V_\wedge, V_\vee^I$ and $V_\wedge^I$ satisfy the following:

i. If $a = \perp$, then $V_\vee = V_\wedge = V_\vee^I = V_\wedge^I = T: L^2 \to L$ where

$$T(x,y) = \begin{cases} T_\alpha(x,y) & if\ (x,y) \in L_\alpha \times L_\alpha, \\ x \wedge y & otherwise. \end{cases}$$

is a t-norm on L i.e. the functions $V_\vee, V_\wedge, V_\vee^I$ and $V_\wedge^I$ are reduced to the lattice-based sum construction of t-norms on L given in [31].

ii. If $a = \top$, then $V_\vee = V_\wedge = V_\vee^I = V_\wedge^I = S: L^2 \to L$ where,

$$S(x,y) = \begin{cases} S_\alpha(x,y) & if\ (x,y) \in L_\alpha \times L_\alpha, \\ x \vee y & otherwise. \end{cases}$$

is a t-conorm on L i.e. the functions $V_\vee, V_\wedge, V_\vee^I$ and $V_\wedge^I$ are reduced to the lattice-based sum construction of t-conorms on L given in [31].

We end this section by showing some examples. For more examples for the construction of t-norms and t-conorms we recommended [31].

Example 4.6:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ in Figure 4-5 and the lattice-based sum of bounded lattices L in Figure 4-6 where $L_{\perp_\Lambda} = \{0, a, b, c\}$, $L_\varepsilon = L_\gamma = \{c\}$, $L_\alpha = \{c, d, e, f\}$, $L_\beta = \{m, n, k, f\}$, $L_\delta = \{c, h, g, r\}$, $L_\mu = \{f\}$ and $L_{\top_\Lambda} = L_\nu = \{1\}$.

Define a drastic product t-norm T_D and a drastic sum t-conorm S_D on $L_{\perp_\Lambda}, L_\alpha, L_\beta$ and L_δ. It is easy to check that the operation T whose values are written in Table 4-11 is a t-norm on L with neutral element 1. Also, the operation S whose values are written in Table 4-12 is a t-conorm on L with neutral element 0.

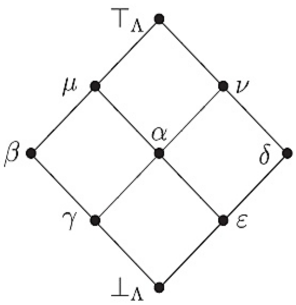

Figure 4-5 The lattice $(\Lambda, \sqsubseteq)$ of Example 4.6

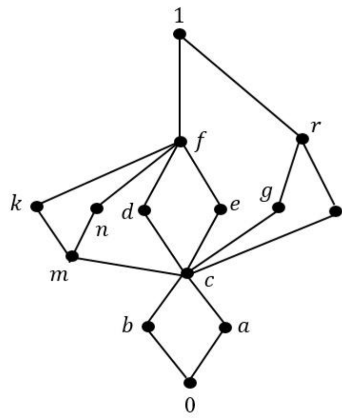

Figure 4-6 The lattice L of Example 4.6

Table 4-11 The t-norm T on L of Example 4.6

T	0	a	b	c	d	e	f	g	h	r	m	n	k	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	a	a	a	a	a	a	a	a	a	a	a
b	0	0	0	b	b	b	b	b	b	b	b	b	b	b
c	0	a	b	c	c	c	c	c	c	c	c	c	c	c
d	0	a	b	c	c	c	d	c	c	c	c	c	c	d
e	0	a	b	c	c	c	e	c	c	c	c	c	c	e
f	0	a	b	c	d	e	f	c	c	c	m	n	k	f
g	0	a	b	c	c	c	c	c	c	g	c	c	c	g
h	0	a	b	c	c	c	c	c	c	h	c	c	c	h
r	0	a	b	c	c	c	c	g	h	r	c	c	c	r
m	0	a	b	c	c	c	m	c	c	c	m	m	m	m
n	0	a	b	c	c	c	n	c	c	c	m	m	m	n
k	0	a	b	c	c	c	k	c	c	c	m	m	m	k
1	0	a	b	c	d	e	f	g	h	r	m	n	k	1

Table 4-12 The t-conorm S on L of Example 4.6

S	0	a	b	c	d	e	f	g	h	r	m	n	k	1
0	0	a	b	c	d	e	f	g	h	r	m	n	k	1
a	a	c	c	c	d	e	f	g	h	r	m	n	k	1
b	b	c	c	c	d	e	f	g	h	r	m	n	k	1
c	c	c	c	c	d	e	f	g	h	r	m	n	k	1
d	d	d	d	d	f	f	f	1	1	1	f	f	f	1
e	e	e	e	e	f	f	f	1	1	1	f	f	f	1
f	f	f	f	f	f	f	f	1	1	1	f	f	f	1
g	g	g	g	g	1	1	1	r	r	r	1	1	1	1
h	h	h	h	h	1	1	1	r	r	r	1	1	1	1
r	r	r	r	r	1	1	1	r	r	r	1	1	1	1
m	m	m	m	m	f	f	f	1	1	1	m	n	k	1
n	n	n	n	n	f	f	f	1	1	1	n	f	f	1
k	k	k	k	k	f	f	f	1	1	1	k	f	f	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

CHAPTER FIVE

LATTICE-BASED SUM CONSTRUCTION OF UNINORMS ON BOUNDED LATTICES

Chapter five
Lattice-based sum construction of uninorms on bounded lattices

5.1 Introduction

In this chapter we give our proposal method for constructing uninorms on bounded lattices. Similarly, as in the case of nullnorm, we also restrict our consideration for the lattice-ordered index set to be finite and each summand of the underlying bounded lattice L to be a bounded lattice. The neutral element e of the uninorm may be equal to one of the boundaries of some summand or inside some summand. Therefore, we restrict our consideration to the case in which e is one of the boundaries of some summand, and we will explain what will happen if it is inside some summand. By our construction methods introduced in this chapter, we can obtain t-norms and t-conorms from a given family of t-norms and t-conorms on bounded lattices. The idempotent uninorms construction on bounded lattices is also available by our construction methods, just by putting the corresponding idempotent t-norm and idempotent t-conorm on each summand lattice of the underlying bounded lattice L.

5.2 Construction of uninorms on bounded lattices

Theorem 5.1:

Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and let $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \bot_\alpha, \top_\alpha)$ be a lattice-based sum of bounded lattices. Let $e \in L$ with $e \in \{\bot_\alpha, \top_\alpha\}$ for some $\alpha \in \Lambda$ and $(T_\alpha)_{\alpha \in \Lambda}$ $((S_\alpha)_{\alpha \in \Lambda})$ be a family of t-norms (t-conorms) on the corresponding summands $(L_\alpha)_{\alpha\in\Lambda}$. Then the functions $U_\downarrow: L^2 \to L$ and $U_\uparrow: L^2 \to L$ defined as follows:

$$U_\downarrow(x,y) = \begin{cases} T_\alpha(x,y) & if\ x,y \in L_\alpha \cap\downarrow e, \\ S_\beta(x,y) & if\ x,y \in L_\beta \cap\uparrow e, \\ x \wedge y & if\ x \in L_\alpha \cap\downarrow e, y \in L_\beta \cap\downarrow e, \alpha \neq \beta, \\ y & if\ x \in\downarrow e\ and\ y \parallel e, \\ x & if\ y \in\downarrow e\ and\ x \parallel e, \\ x \vee y & otherwise. \end{cases} \tag{5.1}$$

and

$$U_\uparrow(x,y) = \begin{cases} T_\alpha(x,y) & if\ x,y \in L_\alpha \cap\downarrow e, \\ S_\beta(x,y) & if\ x,y \in L_\beta \cap\uparrow e, \\ x \vee y & if\ x \in L_\alpha \cap\uparrow e, y \in L_\beta \cap\uparrow e, \alpha \neq \beta, \\ y & if\ x \in\uparrow e\ and\ y \parallel e, \\ x & if\ y \in\uparrow e\ and\ x \parallel e, \\ x \wedge y & otherwise. \end{cases} \tag{5.2}$$

are uninorms on L with neutral element e.

Proof:

The proof runs only for the operation $U_\downarrow$. The operation $U_\uparrow$ has a similar proof.

First it is necessary to check that the operation $U_\downarrow$ is well defined. A problem can only arise if $(x,y) \in L_\alpha \times L_\beta$ with $x \in L_\alpha \cap L_\beta$ for some $\alpha,\beta \in \Lambda$ and write,

i. $x,y \in\downarrow e$,

   a) $\alpha \sqsubset \beta$. In this case: $U_\downarrow(x,y) = T_\beta(x,y) = x \wedge_\beta y = x$ if we consider that $x,y \in L_\beta$ and $U_\downarrow(x,y) = x \wedge y = x$ if we consider that $x \in L_\alpha$ and $y \in L_\beta$. Thus getting the same result in both cases.

b) $\alpha \parallel \beta$. In this case we have either $x = \top_\alpha = \top_\beta$ and hence, $U_\downarrow(x, y) = T_\beta(x, y) = x \wedge_\beta y = x \wedge y = y$ or $x = \bot_\alpha = \bot_\beta$ and hence, $U_\downarrow(x, y) = T_\beta(x, y) = x \wedge_\beta y = x \wedge y = x$.

ii. $x, y \in \uparrow e$. In this case we have a dual proof of Case (i) due to the duality between the t-norm and the t-conorm.

Now, we need to prove that $U_\downarrow$ is a uninorm on L with neutral element e.

Commutativity: The commutativity of $U_\downarrow$ is preserved due to the commutativity of the t-norm and the t-conorm on each summand also $\wedge$ and $\vee$ on L.

Neutral element: We prove that e is the neutral element of $U_\downarrow$ by splitting the following cases for some $x \in L$

i. $x \in \downarrow e$,

a) If there exists some $\alpha \in \Lambda$ such that $\{x, e\} \subseteq L_\alpha$, then (from Remark 4.1) we have,

$$U_\downarrow(x, e) = T_\alpha(x, e) = x \wedge_\alpha e = x$$

b) If there is no $\alpha \in \Lambda$ such that $\{x, e\} \subseteq L_\alpha$, then

$$U_\downarrow(x, e) = x \wedge e = x$$

ii. $x \in \uparrow e$. This case has a dual proof of Case (i) due to the duality between the t-norm and the t-conorm.

iii. $x \parallel e$. Then directly from the definition of $U_\downarrow$ we have

$$U_\downarrow(x, e) = x$$

Monotonicity: We prove that if $x \leq y$ then for all $z \in L$, $U_{\downarrow}(x,z) \leq U_{\downarrow}(y,z)$. The proof is split into all the possible cases, as follows:

Case (1): Suppose that $x, y \in \downarrow e$. Then we have the following subcases

Subcase 1(a): $z \in \downarrow e$,

i. There exists some $\alpha \in \Lambda$ such that $\{x, y\} \subseteq L_{\alpha}$. If $z \in L_{\alpha}$, then monotonicity holds trivially due to the monotonicity of T_{α} on L_{α}. If $z \notin L_{\alpha}$, then
$$U_{\downarrow}(x,z) = x \wedge z \leq y \wedge z = U_{\downarrow}(y,z)$$

ii. If there is no $\alpha \in \Lambda$ such that $\{x, y\} \subseteq L_{\alpha}$, then we have one of the following possibilities

   a) x and z are in the same summand. We observe it by considering $\{x, z\} \subseteq L_{\alpha}$ and $y \notin L_{\alpha}$, then
   $$U_{\downarrow}(x,z) = T_{\alpha}(x,z) \leq x \wedge z \leq y \wedge z = U_{\downarrow}(y,z)$$

   b) If y and z are in the same summand, we observe it by considering $\{y, z\} \subseteq L_{\alpha}$ and $x \in L_{\beta}$ with $\alpha \neq \beta$ for some $\alpha, \beta \in \Lambda$. Then (from Lemma 4.1) we have either $\beta \sqsubset \alpha$ or $\beta \parallel \alpha$. In case $\beta \sqsubset \alpha$ then $x < u$ for all $u \in L_{\alpha}$ and hence, $U_{\downarrow}(x,z) = x \wedge z = x \leq T_{\alpha}(y,z) = U_{\downarrow}(y,z)$
   In case $\beta \parallel \alpha$, we have one of the following possibilities

      i) $y \in \{\perp_{\alpha}, \top_{\alpha}\}$, and hence we have
      $$U_{\downarrow}(x,z) = x \wedge z \leq y \wedge z = T_{\alpha}(y,z) = U_{\downarrow}(y,z)$$

      ii) $y \in L_{\alpha} \backslash \{\perp_{\alpha}, \top_{\alpha}\}$, then necessarily $x = \perp_{\beta}$ and hence we have $U_{\downarrow}(x,z) = x \wedge z = x \leq T_{\alpha}(y,z) = U_{\downarrow}(y,z)$

iii. All arguments are in different summands. Then monotonicity holds due to the monotonicity of $\wedge$ on L.

Subcase 1(b): $z \in\uparrow e \Longrightarrow U_{\downarrow}(x,z) = x \vee z = z = y \vee z = U_{\downarrow}(y,z)$

Subcase 1(c): $z \parallel e \Longrightarrow U_{\downarrow}(x,z) = z = U_{\downarrow}(y,z)$

Case (2): Suppose that $x, y \in\uparrow e$. Then we have the following subcases

Subcase 2(a): $z \in\uparrow e$. This case has a duel proof of Case 1(a) due to the duality between the t-norm and the t-conorm.

Subcase 2(b): If $z \in\downarrow e$ or $z \parallel e$, then

$$U_{\downarrow}(x,z) = x \vee z \leq y \vee z = U_{\downarrow}(y,z)$$

Case (3): Suppose that $x \in\downarrow e, y \in\uparrow e$,

i. $z \in\downarrow e$. Then we have either x and z are in the same summand or x and z are in different summands. In both cases and due to the t-norm on each summand and $\wedge$ on L we have $U_{\downarrow}(x,z) \leq e$ and hence,

$$U_{\downarrow}(x,z) \leq e \leq y = y \vee z = U_{\downarrow}(y,z)$$

ii. If $z \in\uparrow e$, then we have either y and z are in the same summand or y and z are in different summands. In both cases and due to the t-conorm defined on each summand and $\vee$ on L we have $z \leq U_{\downarrow}(y,z)$ and hence we have

$$U_{\downarrow}(x,z) = x \vee z = z \leq U_{\downarrow}(y,z)$$

iii. $z \parallel e$, then $U_{\downarrow}(x,z) = z \leq y \vee z = U_{\downarrow}(y,z)$

Case (4): Suppose that $x \in\downarrow e, y \parallel e$,

i. If $z \in\downarrow e$, then in similar way of Case 3(i), we have $U_{\downarrow}(x,z) \leq x$ and hence,

$$U_{\downarrow}(x,z) \leq x \leq y = U_{\downarrow}(y,z)$$

ii. $z \in \uparrow e \Longrightarrow U_{\downarrow}(x,z) = x \vee z \leq y \vee z = U_{\downarrow}(y,z)$

iii. $z \parallel e \Longrightarrow U_{\downarrow}(x,z) = z \leq y \vee z = U_{\downarrow}(y,z)$

Case (5): Suppose that $x \parallel e, y \in \uparrow e$.

i. If $z \in \downarrow e$, then $U_{\downarrow}(x,z) = x \leq y = y \vee z = U_{\downarrow}(y,z)$

ii. If $z \in \uparrow e$, then $U_{\downarrow}(x,z) = x \vee z \leq y \vee z \leq U_{\downarrow}(y,z)$

iii. If $z \parallel e$, then $U_{\downarrow}(x,z) = x \vee z \leq y \vee z = U_{\downarrow}(y,z)$

Case (6): Suppose that $x \parallel e, y \parallel e$,

i. $z \in \downarrow e$, then $U_{\downarrow}(x,z) = x \leq y = U_{\downarrow}(y,z)$

ii. $z \in \uparrow e$ or $z \parallel e$, then $U_{\downarrow}(x,z) = x \vee z \leq y \vee z = U_{\downarrow}(y,z)$

Associativity: We prove that $U_{\downarrow}\big(x, U_{\downarrow}(y,z)\big) = U_{\downarrow}(U_{\downarrow}(x,y),z)$ for all $x,y,z \in L$. Again, the proof is split into all the possible cases by considering the relationship between the arguments x,y,z and e as follows:

Case (1): Suppose that all arguments are from $\downarrow e$. Then we have one of the following possibilities

i. There exists some $\alpha \in \Lambda$ such that $\{x,y,z\} \subseteq L_{\alpha}$. In this case the associativity holds trivially due to the associativity of T_{α} on L_{α}.

ii. All arguments are from different summands. In this case the associativity holds trivially due to the associativity of $\wedge$ on L.
In this case, we must note that, if $x \wedge y$ and z are in the same summand, then necessarily $x \wedge y$ is equal to one of the boundaries of this summand and hence the associativity holds due to the associativity of $\wedge$.

iii. Exactly two arguments are from the same summand. We observe it by considering the following cases:

a) There exist some $\alpha \in \Lambda$ such that $\{x, y\} \subseteq L_\alpha$ and $z \notin L_\alpha$. If x or y is equal to one of the boundaries of L_α then associativity holds trivially due to the associativity of $\wedge$ on L. Therefore, we assume that $x, y \in L_\alpha \backslash \{\bot_\alpha, \top_\alpha\}$. In this case, we compare z with x and y, as follows

If $x > z$ or $y > z$, then

$$U_\downarrow(U_\downarrow(x, y), z) = U_\downarrow(T_\alpha(x, y), z) = T_\alpha(x, y) \wedge z = z$$

$$U_\downarrow\big(x, U_\downarrow(y, z)\big) = U_\downarrow(x, y \wedge z) = U_\downarrow(x, z) = x \wedge z = z$$

If $x < z$ or $y < z$, then

$$\begin{aligned} U_\downarrow(U_\downarrow(x, y), z) = U_\downarrow(T_\alpha(x, y), z) &= T_\alpha(x, y) \wedge z \\ &= T_\alpha(x, y) \end{aligned}$$

$$U_\downarrow\big(x, U_\downarrow(y, z)\big) = U_\downarrow(x, y \wedge z) = U_\downarrow(x, y) = T_\alpha(x, y)$$

If $x \parallel z$ or $y \parallel z$, then $x \wedge z = y \wedge z = x \wedge y \wedge z = T_\alpha(x, y) \wedge z$ and hence,

$$\begin{aligned} U_\downarrow(U_\downarrow(x, y), z) = U_\downarrow(T_\alpha(x, y), z) &= T_\alpha(x, y) \wedge z \\ &= x \wedge y \wedge z = x \wedge U_\downarrow(y, z) \\ &= U_\downarrow\big(x, U_\downarrow(y, z)\big) \end{aligned}$$

b) There exists some $\beta \in \Lambda$ such that $\{x, z\} \subseteq L_\beta$ and $y \notin L_\beta$. This case is similar to Case (a) resulting in similar proof.

c) There exists some $\delta \in \Lambda$ such that $\{y, z\} \subseteq L_\delta$ and $x \notin L_\delta$. This case is similar to Case (a) resulting in similar proof.

Case (2): Suppose that all arguments are from $\uparrow e$. This case has a dual proof to Case (1) due to the duality between the t-norm and the t-conorm operators.

Case (3): Suppose that exactly two arguments are from $\downarrow e$. We observe it by distinguishing the following subcases

Subcase 3(a): Assume that $x, y \in \downarrow e$ and $z \notin \downarrow e$. In this case we have either x and y are in the same summand or x and y are in different summands. In both cases, we have $U_\downarrow(x, y) \leq e$ and hence for

i. $z > e$,

$$\begin{aligned} U_\downarrow\big(x, U_\downarrow(y, z)\big) &= U_\downarrow(x, y \vee z) = U_\downarrow(x, z) \\ &= x \vee z = z = U_\downarrow(x, y) \vee z \\ &= U_\downarrow(U_\downarrow(x, y), z) \end{aligned}$$

ii. $z \parallel e$, $U_\downarrow\big(x, U_\downarrow(y, z)\big) = U_\downarrow(x, z) = z = U_\downarrow(U_\downarrow(x, y), z)$

Subcase 3(b): Assume that $x, z \in \downarrow e$ and $y \notin \downarrow e$. This case is similar to Case 3(a) resulting in similar proof.

Subcase 3(c): Assume that $y, z \in \downarrow e$ and $x \notin \downarrow e$. This case is similar to Case 3(a) resulting in similar proof.

Case (4): Suppose that exactly two arguments are from $\uparrow e$. We observe it by distinguishing the following subcases

Subcase 4(a): Assume that $x, y \in \uparrow e$ and $z \notin \uparrow e$. In this case we have either x and y are in the same summand or x and y are in different summands. In both cases, we have $U_\downarrow(x, y) \geq e$ and hence for

i. $z < e$,

$$\begin{aligned} U_\downarrow\big(x, U_\downarrow(y, z)\big) &= U_\downarrow(x, y \vee z) = U_\downarrow(x, y) \\ &= U_\downarrow(x, y) \vee z = U_\downarrow(U_\downarrow(x, y), z) \end{aligned}$$

ii. If $z \parallel e$, then $U_\downarrow(y, z) = y \vee z \geq e$ and hence we distinguish the following cases

a) There exists some $\beta \in \Lambda$ such that $\{x, y\} \subseteq L_\beta$, then we have either $y \vee z \in L_\beta$ or $y \vee z \notin L_\beta$. In case $y \vee z \in L_\beta$ then associativity holds due to the associativity of S_β on L_β. In case $y \vee z \notin L_\beta$ the associativity holds due to the associativity of $\vee$ on L.

b) There is no $\beta \in \Lambda$ such that $\{x, y\} \subseteq L_\beta$, then associativity holds due to the associativity of $\vee$ on L.

Subcase 4(b): Assume that $x, z \in \uparrow e$ and $y \notin \uparrow e$. In this case, we have either x and z are in the same summand or x and z are in different summands. In both cases, we have

i. $y < e$,

$$\begin{aligned} U_\downarrow\big(x, U_\downarrow(y, z)\big) &= U_\downarrow(x, y \vee z) = U_\downarrow(x, z) \\ &= U_\downarrow(x \vee y, z) \\ &= U_\downarrow(U_\downarrow(x, y), z) \end{aligned}$$

ii. $y \parallel e$. This case is similar to *subcase* 4(a (ii)) resulting in similar proof.

Subcase 4(c): Assume that $y, z \in \uparrow e$ and $x \notin \uparrow e$. In this case, we have either y and z are in the same summand or y and z are in different summands. In both cases, we have $U_\downarrow(y, z) \geq e$ and hence for,

i. $x < e$, we have

$$\begin{aligned} U_\downarrow\big(x, U_\downarrow(y, z)\big) &= x \vee U_\downarrow(y, z) = U_\downarrow(y, z) \\ &= U_\downarrow(x \vee y, z) = U_\downarrow(U_\downarrow(x, y), z) \end{aligned}$$

ii. $x \parallel e$. This case is similar to *subcase* 4(a(ii)) resulting in similar proof.

Case (5): Suppose that exactly two arguments are incomparable with e. We observe it by distinguishing the following subcases

Subcase 5(a): Assume that $x \parallel e, y \parallel e$ and $z \nparallel e$. Then we have $U_{\downarrow}(x,y) = x \vee y$ and hence we have one of the following possibilities

i. $x \vee y > e$,

a) $z < e$,

$$U_{\downarrow}\big(x, U_{\downarrow}(y,z)\big) = U_{\downarrow}(x,y) = x \vee y = x \vee y \vee z$$
$$= U_{\downarrow}(x,y) \vee z = U_{\downarrow}(U_{\downarrow}(x,y), z)$$

b) $z > e$. In this case we have either z and $x \vee y$ are in the same summand or z and $x \vee y$ are in different summands. In both cases and from the fact that $x \vee y$ is necessarily on the boundaries for some summand, we have

$$U_{\downarrow}\big(x, U_{\downarrow}(y,z)\big) = U_{\downarrow}(x, y \vee z) = x \vee y \vee z$$
$$= U_{\downarrow}(x,y) \vee z = U_{\downarrow}(U_{\downarrow}(x,y), z)$$

ii. $x \vee y \parallel e$,

a) $z < e$,

$$U_{\downarrow}\big(x, U_{\downarrow}(y,z)\big) = U_{\downarrow}(x,y) = x \vee y$$
$$= U_{\downarrow}(x \vee y, z) = U_{\downarrow}(U_{\downarrow}(x,y), z)$$

b) $z > e$,

$$U_{\downarrow}\big(x, U_{\downarrow}(y,z)\big) = U_{\downarrow}(x, y \vee z) = x \vee y \vee z$$
$$= U_{\downarrow}(x,y) \vee z = U_{\downarrow}(U_{\downarrow}(x,y), z)$$

Subcase 5(b): $x \parallel e, z \parallel e$ and $y \nparallel e$, this case is similar to *subcase* 5(a) resulting in a similar proof.

Subcase 5(c): $y \parallel e, z \parallel e$ and $x \nparallel e$, this case is similar to *subcase* 5(a) resulting in a similar proof.

For the remaining possibilities we distinguish the following cases

i. $x \in \downarrow e, y \in \uparrow e, z \parallel e,$

$$U_\downarrow\big(x, U_\downarrow(y,z)\big) = U_\downarrow(x, y \vee z) = x \vee y \vee z = y \vee z = U_\downarrow(y,z)$$
$$= U_\downarrow(x \vee y, z) = U_\downarrow(U_\downarrow(x,y), z)$$

ii. $x \in \downarrow e, y \parallel e, z \in \uparrow e,$

$$U_\downarrow\big(x, U_\downarrow(y,z)\big) = U_\downarrow(x, y \vee z) = x \vee y \vee z = y \vee z$$
$$= U_\downarrow(y,z) = U_\downarrow(U_\downarrow(x,y), z)$$

iii. $x \in \uparrow e, y \in \downarrow e, z \parallel e,$

$$U_\downarrow\big(x, U_\downarrow(y,z)\big) = U_\downarrow(x,z) = U_\downarrow(x \vee y, z) = U_\downarrow(U_\downarrow(x,y), z)$$

iv. $x \in \uparrow e, y \parallel e, z \in \downarrow e,$

$$U_\downarrow\big(x, U_\downarrow(y,z)\big) = U_\downarrow(x,y) = U_\downarrow(x,y) \vee z = U_\downarrow(U_\downarrow(x,y), z)$$

v. $x \parallel e, y \in \downarrow e, z \in \uparrow e,$

$$U_\downarrow\big(x, U_\downarrow(y,z)\big) = U_\downarrow(x, y \vee z) = U_\downarrow(x,z) = U_\downarrow(U_\downarrow(x,y), z)$$

vi. $x \parallel e, y \in \uparrow e, z \in \downarrow e,$

$$U_\downarrow\big(x, U_\downarrow(y,z)\big) = U_\downarrow(x, y \vee z) = U_\downarrow(x,y) = U_\downarrow(x,y) \vee z$$
$$= U_\downarrow(U_\downarrow(x,y), z)$$

Example 5.1:

Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ in Figure 5-1 and its lattice-based sum of bounded lattices L in Figure 5-2 where $L_{\perp_\Lambda} = \{0, a, b, c\}$, $L_\delta = \{c, d\}$, $L_\beta = \{d, f, g, h\}$, $L_\alpha = \{e, h\}$ and $L_{\top_\Lambda} = \{h, m, n, 1\}$. Let $T_\alpha = T_D$ and $S_\alpha = S_D$ on L_α for all $\alpha \in \Lambda$. Then the functions $U_\downarrow$ and $U_\uparrow$ whose values are written in Tables 5-1 and 5-2 are uninorms on L with neutral element e which are constructed using Equations (5.1) and (5.2), respectively.

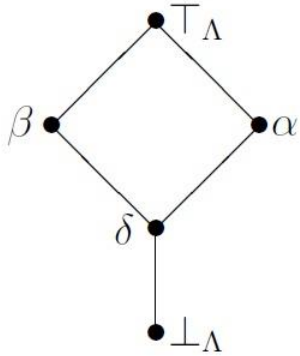

Figure 5-1 The lattice $(\Lambda, \sqsubseteq)$ of Example 5.1

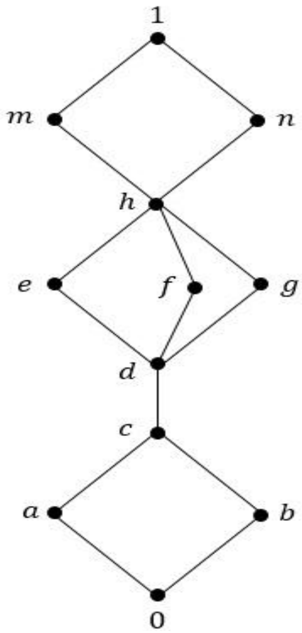

Figure 5-2 The lattice L of Example 5.1

Table 5-1 The uninorm $U_{\downarrow}$ on L of Example 5.1

$U_{\downarrow}$	0	a	b	c	d	f	g	h	m	n	e	1
0	0	0	0	0	0	f	g	h	m	n	0	1
a	0	0	0	a	a	f	g	h	m	n	a	1
b	0	0	0	b	b	f	g	h	m	n	b	1
c	0	a	b	c	c	f	g	h	m	n	c	1
d	0	a	b	c	d	f	g	h	m	n	d	1
f	f	f	f	f	f	f	h	h	m	n	f	1
g	g	g	g	g	g	h	g	h	m	n	g	1
h	h	h	h	h	h	h	h	h	m	n	h	1
m	m	m	m	m	m	m	m	m	1	1	m	1
n	n	n	n	n	n	n	n	n	1	1	n	1
e	0	a	b	c	d	f	g	h	m	n	e	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Table 5-2 The uninorm $U_{\uparrow}$ on L of Example 5.1

$U_{\uparrow}$	0	a	b	c	d	f	g	h	m	n	e	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	a	a	a	a	a	a	a	a	a
b	0	0	0	b	b	b	b	b	b	b	b	b
c	0	a	b	c	c	c	c	c	c	c	c	c
d	0	a	b	c	d	d	d	d	d	d	d	d
f	0	a	b	c	d	f	d	f	f	f	f	f
g	0	a	b	c	d	d	g	g	g	g	g	g
h	0	a	b	c	d	f	g	h	m	n	h	1
m	0	a	b	c	d	f	g	m	1	1	m	1
n	0	a	b	c	d	f	g	n	1	1	n	1
e	0	a	b	c	d	f	g	h	m	n	e	1
1	0	a	b	c	d	f	g	1	1	1	1	1

Corollary 5.1:

With all the assumptions of Theorem 5.1, the uninorm functions $U_{\downarrow}$ and $U_{\uparrow}$ as defined in Equations (5.1) and (5.2), respectively, satisfy the following:

i. If $e = \top$, then

$$U_{\downarrow}(x,y) = U_{\uparrow}(x,y) = T(x,y) = \begin{cases} T_{\alpha}(x,y) & if\ (x,y) \in L_{\alpha} \times L_{\alpha}, \\ x \wedge y & otherwise. \end{cases}$$

is a t-norm on L i.e. the functions $U_{\downarrow}$ and $U_{\uparrow}$ are reduced to the lattice-based sum construction of t-norms on L given in [31].

ii. If $e = \perp$, then

$$U_{\downarrow}(x,y) = U_{\uparrow}(x,y) = S(x,y) = \begin{cases} S_{\alpha}(x,y) & if\ (x,y) \in L_{\alpha} \times L_{\alpha}, \\ x \vee y & otherwise. \end{cases}$$

is a t-conorm on L i.e. the functions $U_{\downarrow}$ and $U_{\uparrow}$ are reduced to the lattice-based sum construction of t-conorms on L given in [31].

Remark 5.1:

The obtained results in Corollary 5.1 are special cases of the obtained general results in [31], in which it depends on t-subnorms (t-subconorms by duality) and a lattice-ordered index set which need not be finite.

Remark 5.2:

Given a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha \in \Lambda} (L_{\alpha}, \leq_{\alpha}, \perp_{\alpha}, \top_{\alpha})$, then the functions $U_{\downarrow}$ and $U_{\uparrow}$ in Theorem 5.1 are disjunctive and conjunctive uninorms on L, respectively, such that $U_{\downarrow}(\perp, \top) = \top$ and $U_{\uparrow}(\perp, \top) = \perp$.

Corollary 5.2:

If we put $T_{\alpha} = T_M^L$ and $S_{\alpha} = S_M^L$ on L_{α} for all $\alpha \in \Lambda$ in Equation (5.1) and (5.2) in Theorem 5.1, then the functions $U_{\downarrow}^D$ and $U_{\uparrow}^C$ defined as

$$U_{\downarrow}^{D}(x,y)=\begin{cases}x\wedge y & if\ x,y\in\downarrow e,\\ y & if\ x\in\downarrow e\ and\ y\parallel e,\\ x & if\ y\in\downarrow e\ and\ x\parallel e,\\ x\vee y & otherwise.\end{cases} \tag{5.3}$$

and

$$U_{\uparrow}^{C}(x,y)=\begin{cases}x\vee y & if\ x,y\in\uparrow e,\\ y & if\ x\in\uparrow e\ and\ y\parallel e,\\ x & if\ y\in\uparrow e\ and\ x\parallel e,\\ x\wedge y & otherwise.\end{cases} \tag{5.4}$$

are *disjunctive* and *conjunctive idempotent* uninorms on L, respectively.

Note that, the obtained uninorms $U_{\downarrow}^{D}$ and $U_{\uparrow}^{C}$ in Equations (5.3) and (5.4), respectively, are exactly the same greatest and smallest idempotent uninorms, respectively, obtained in [11].

Example 5.2:

Consider the lattice-ordered index set $(\Lambda,\sqsubseteq)$ and the lattice-based sum of bounded lattice L of Example 5.1. Let $T_{\alpha}=T_{M}^{L}$ and $S_{\alpha}=S_{M}^{L}$ on L_{α} for all $\alpha\in\Lambda$. Then the functions $U_{\downarrow}^{D}$ and $U_{\uparrow}^{C}$ whose values are written in Tables 5-3 and 5-4 are, respectively, idempotent uninorms on L with neutral element e.

Table 5-3 The idempotent uninorm $U_{\downarrow}^{D}$ on L of Example 5.2

$U_{\downarrow}^{D}$	0	a	b	c	d	f	g	h	m	n	e	1
0	0	0	0	0	0	f	g	h	m	n	0	1
a	0	a	0	a	a	f	g	h	m	n	a	1
b	0	0	b	b	b	f	g	h	m	n	b	1
c	0	a	b	c	c	f	g	h	m	n	c	1
d	0	a	b	c	d	f	g	h	m	n	d	1
f	f	f	f	f	f	f	h	h	m	n	f	1
g	g	g	g	g	g	h	g	h	m	n	g	1
h	h	h	h	h	h	h	h	h	m	n	h	1
m	m	m	m	m	m	m	m	m	m	1	m	1
n	n	n	n	n	n	n	n	n	1	n	n	1
e	0	a	b	c	d	f	g	h	m	n	e	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Table 5-4 The idempotent uninorm $U_{\uparrow}^{C}$ on L of Example 5.2

$U_{\uparrow}^{C}$	0	a	b	c	d	f	g	h	m	n	e	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a	a	a	a	a
b	0	0	b	b	b	b	b	b	b	b	b	b
c	0	a	b	c	c	c	c	c	c	c	c	c
d	0	a	b	c	d	d	d	d	d	d	d	d
f	0	a	b	c	d	f	d	f	f	f	f	f
g	0	a	b	c	d	d	g	g	g	g	g	g
h	0	a	b	c	d	f	g	h	m	n	h	1
m	0	a	b	c	d	f	g	m	m	1	m	1
n	0	a	b	c	d	f	g	n	1	n	n	1
e	0	a	b	c	d	f	g	h	m	n	e	1
1	0	a	b	c	d	f	g	1	1	1	1	1

Corollary 5.3:

If we put $T_{\alpha} = T_{D}^{L}$ and $S_{\alpha} = S_{D}^{L}$ on L_{α} for all $\alpha \in \Lambda$ in Equation (5.1) and (5.2) in Theorem 5.1, then we obtain the following uninorms on L with neutral element $e \in L$:

$$U_{\downarrow}^{d}(x,y)=\begin{cases}\bot_{\alpha} & if\ x,y\in(L_{\alpha}\cap\downarrow e)\backslash\{\top_{\alpha}\},\\ \top_{\beta} & if\ x,y\in\left(L_{\beta}\cap\uparrow e\right)\backslash\{\bot_{\beta}\},\\ x\wedge y & if\ x\in L_{\alpha}\cap\downarrow e,y\in L_{\beta}\cap\downarrow e,\alpha\neq\beta,\\ y & if\ x\in\downarrow e\ and\ y\parallel e,\\ x & if\ y\in\downarrow e\ and\ x\parallel e,\\ x\vee y & otherwise.\end{cases}$$

$$U_{\uparrow}^{d}(x,y)=\begin{cases}\bot_{\alpha} & if\ x,y\in(L_{\alpha}\cap\downarrow e)\backslash\{\top_{\alpha}\},\\ \top_{\beta} & if\ x,y\in(L_{\beta}\cap\uparrow e)\backslash\{\bot_{\beta}\},\\ x\vee y & if\ x\in L_{\alpha}\cap\uparrow e,y\in L_{\beta}\cap\uparrow e,\alpha\neq\beta,\\ y & if\ x\in\uparrow e\ and\ y\parallel e,\\ x & if\ y\in\uparrow e\ and\ x\parallel e,\\ x\wedge y & otherwise.\end{cases}$$

Remark 5.3:

Given a lattice-ordered index set $(\Lambda,\sqsubseteq)$ and a lattice-based sum of bounded lattices $L=\bigoplus_{\alpha\in\Lambda}(L_{\alpha},\leq_{\alpha},\bot_{\alpha},\top_{\alpha})$ and $e\in L$, then the functions $U_{\downarrow}$ and $U_{\uparrow}$ in Theorem 5.1 can be equivalently defined as

$$U_{\downarrow}(x,y)=\begin{cases}T_{e}(x,y) & if\ x,y\in\downarrow e,\\ S_{e}(x,y) & if\ x,y\in\uparrow e,\\ H(x)\vee H(y) & otherwise.\end{cases}$$

$$U_{\uparrow}(x,y)=\begin{cases}T_{e}(x,y) & if\ x,y\in\downarrow e,\\ S_{e}(x,y) & if\ x,y\in\uparrow e,\\ M(x)\wedge M(y) & otherwise.\end{cases}$$

where $H,M:L^{2}\to L$ are mappings given by

$$H(x)=\begin{cases}\bot & if\ x\in\downarrow e,\\ x & otherwise.\end{cases},M(x)=\begin{cases}\top & if\ x\in\uparrow e,\\ x & otherwise.\end{cases}$$

and T_{e},S_{e} are lattice-based sum of t-norms and t-conorms on $\downarrow e$ and $\uparrow e$, respectively, as a direct consequence from [31].

Remark 5.4:

The uninorms $U_\downarrow$ and $U_\uparrow$ given in Equations (5.1) and (5.2), respectively are based on a family of t-norms and t-conorms defined on each summand lattice of the underlying lattice-based sum L. Consequently, given a bounded lattice $(L, \leq, \perp, \top)$ and $e \in L$ and a t-norm T_e on $[\perp, e]$ and a t-conorm S_e on $[e, \top]$ which are not a lattice-based sums, then the functions $U_\downarrow$ and $U_\uparrow$ given in Equations (5.1) and (5.2) are not a uninorms on L as we can see in the following example.

Example 5.3:

Consider the bounded lattice L in Figure 5-2. Let $T_e = T_D^L$ on $[0, e]$ and $S_e = S_D^L$ on $[e, 1]$. Then the functions $U_\downarrow$ and $U_\uparrow$ in Equations (5.1) and (5.2) are not uninorms on L with neutral element e, such that, if the elements $h, f, g \in L$, then we have,

$$U_\downarrow\big(h, U_\downarrow(f, g)\big) = U_\downarrow(h, f \vee g) = U_\downarrow(h, h) = S_e(h, h) = 1,$$

$$U_\downarrow(U_\downarrow(h, f), g) = U_\downarrow(h \vee f, g) = U_\downarrow(h, g) = h \vee g = h.$$

Since $1 \neq h$, then $U_\downarrow$ is not associative and hence $U_\downarrow$ is not a uninorm on L.

Similarly, if we consider the elements $d, f, g \in L$, then we have,

$$U_\uparrow\big(d, U_\uparrow(f, g)\big) = U_\uparrow(d, f \wedge g) = U_\uparrow(d, d) = T_e(d, d) = 0,$$

$$U_\uparrow(U_\uparrow(d, f), g) = U_\uparrow(d \wedge f, g) = U_\uparrow(d, g) = d \wedge g = d.$$

Since $0 \neq d$, then $U_\uparrow$ is not associative and hence $U_\uparrow$ is not a uninorm on L.

Remark 5.5:

The neutral element e of the uninorms $U_\downarrow$ and $U_\uparrow$ in Theorem 5.1 were restricted to be one of the boundaries of some summand lattice of the underlying bounded lattice L. If the neutral element e is inside some summand, then the functions $U_{\downarrow}$ and $U_{\uparrow}$ may not work to construct uninorms on L. For example, if we consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \bot_\alpha, \top_\alpha)$ and there exists some $\alpha \in \Lambda$ such that $\{x, y, e\} \subseteq L_\alpha$ with $\bot_\alpha < x < e < y < \top_\alpha$ and $T_\alpha = T_D^L$, $S_\alpha = S_D^L$ then from Theorem 5.1, we have

$$U_{\downarrow}(x, e) = U_{\uparrow}(x, e) = T_\alpha(x, e) = T_D^L(x, e) = \bot_\alpha \neq x,$$

$$U_{\downarrow}(y, e) = U_{\uparrow}(y, e) = S_\alpha(y, e) = S_D^L(y, e) = \top_\alpha \neq y.$$

This violates the neutral element property of the uninorm operator. However, the functions $U_{\downarrow}$ and $U_{\uparrow}$ are still uninorms on L in case e is inside some summand if and only if the t-norm and the t-conorm defined on this summand are fixed to be the minimum T_M^L and the maximum S_M^L, respectively.

CHAPTER SIX

Chapter six
Conclusions and future work

6.1 Conclusions

In this thesis, based on the lattice-based sum scheme that has been recently introduced by El-Zekey *et al* [30]; we developed new methods for constructing nullnorms and uninorms on bounded lattices which are a lattice-based sum of their summand sublattices. Subsequently, the obtained results are applied for building several new nullnorm and uninorm operations on bounded lattices. As a by-product, the lattice-based sum constructions of t-norms and t-conorms obtained by El-Zekey (see [31]) are obtained in a more general setting where the lattice-ordered index set need not to be finite and so-called t-subnorms (t-subconorms) can be used (with a little restriction) instead of t-norms (t-conorms) as summands. Furthermore, new idempotent nullnorms on bounded lattices, different from the ones given in [16], have been also obtained. It is pointed out that, unlike [16], in our construction of the idempotent nullnorms, the underlying lattices need not to be distributive. We remark that lattice-based sum constructions of non-commutative associative aggregation operators such as pseudo-t-norms, pseudo-t-conorms, pseudo uninorms and pseudo nullnorms can be also obtained just by eliminating the commutativity property.

6.2 Future work

Our work in this thesis open a new gates for the investegation of aggregation functions on bounded lattices. Thus, in the same approach, we can generate other aggregation functions on bounded lattices.

Clearly, inspired by ideas of clifford [17] (in the context of ordinal sums of abstract semigroups), the lattice-based sum approach could deal with lattice-based sums of semigroups.

Note that (see [30]) though a consecutive repetition of standard ordinal and horizontal sum constructions is covered by the lattice-based sum approach, the opposite is not true. First of all, the lattice-based sum can deal also with unbounded posets what is not the case of horizontal sums. Next, the consecutive repetition of mentioned classical construction has impact on the structure of the lattice-ordered index set. These considerations would inevitably lead one into studying the expressive power of lattice-based sums.

APPENDICES

APPENDICES

Appendix A: Python code for test the associativity of Example 3.2

Code:

```
from itertools import combinations_with_replacement

from sympy.abc import a,b,c,d


# Functions definations

T1 = {    (0,0):0, (0,d):0, (0,a):0, (0,b):0, (0,c):0, (0,1):0,
(d,0):0, (d,d):d, (d,a):d, (d,b):d, (d,c):d, (d,1):d, (a,0):0,
(a,d):d, (a,a):a, (a,b):d, (a,c):a, (a,1):a, (b,0):0, (b,d):d,
(b,a):d, (b,b):b, (b,c):b, (b,1):b, (c,0):0, (c,d):d, (c,a):a,
(c,b):b, (c,c):b, (c,1):c, (1,0):0, (1,d):d, (1,a):a, (1,b):b,
(1,c):c, (1,1):1}

def is Associative(T,args=[0,d,a,b,c,1]):

assert len(args) >2

for perm_i in combinations_with_replacement(args,3):

        if T[(perm_i[0],T[(perm_i[1],perm_i[2])])] !=
T[(T[(perm_i[0],perm_i[1])],perm_i[2])]:

            return False, perm_i

    return True, None


def main():

    print is Associative(T1)


if __name__==”__main__”:

    main()
```

Output:

```
(False, (a, c, c))
```

Appendix B: Python code for test the associativity of Example 3.4

Code:

```
from itertools import combinations_with_replacement

from sympy.abc import x,b,c,d

# Functions definations

T1 = {   (0,0):0, (0,c):0, (0,d):0, (0,b):0, (0,x):0, (0,1):0,
(c,0):0, (c,c):c, (c,d):0, (c,b):c, (c,x):c, (c,1):c, (d,0):0,
(d,c):0, (d,d):0, (d,b):d, (d,x):0, (d,1):d, (b,0):0, (b,c):c,
(b,d):d, (b,b):b, (b,x):c, (b,1):b, (x,0):0, (x,c):c, (x,d):0,
(x,b):c, (x,x):x, (x,1):x, (1,0):0, (1,c):c, (1,d):d, (1,b):b,
(1,x):x, (1,1):1}

def is Associative(T,args=[0,x,b,c,d,1]):

    assert len(args) >2

    for perm_i in combinations_with_replacement(args,3):

        if T[(perm_i[0],T[(perm_i[1],perm_i[2])])] !=
T[(T[(perm_i[0],perm_i[1])],perm_i[2])]:

            return False, perm_i

    return True, None

def main():

    print is Associative(T1)

if __name__=="__main__":

main()
```

Output:

```
(True, None)
```

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الملخص باللغة العربية

عمليات التجميع الدامجة على الترتيب الشبكى المحدود هى نوع خاص من عمليات التجميع التى تكون مفيدة فى مجالات كثيرة مثل المنطق الضبابى، والأنظمة الخبيرة، والشبكات العصبية، واستخراج البيانات، ونمذجة النظام الغامض. العمليات من النوع T وعمليات أخرى كثيرة تنتمى إلى فصل عمليات التجميع الدامجة. واحده من الطرق النموذجية فى بناء عمليات التجميع الدامجة على فترة الوحده هى طريقة الجمع الخطى. كما لوحظ سابقا، فى العموم، طريقة الجمع الخطى تفشل غالباً على الترتيب الشبكى. وبناءاً على الملاحظة الأخيرة، حديثا تم إقتراح طريقة جمع جديدة تسمى الجمع المعتمد على الترتيب الشبكى. فى هذه الطريقة المقترحة كان التركيز على الترتيب سواء كان جزئياً أو شبكياً. فى هذه الرسالة، إعتماداً على طريقة الجمع المعتمد على الترتيب الشبكى المحدود، تم إقتراح طرق بناء جديدة لتكوين أهم وأشهر عمليات التجميع الدامجة، ألا وهى العمليات nullnorms و uninorms على الترتيب الشبكى. بعد ذلك، قمنا بتطبيق طرق البناء الجديدة لبناء العديد من العائلات الجديدة من nullnorms و uninorms على الترتيب الشبكى. وكنتيجة ثانوية، تم الحصول على طريقة جديدة لتكوين العمليات t-norms و t-conorms على الترتيب الشبكى.

علاوة على ذلك، تم الحصول على طريقة جديدة لتكوين العملية idempotent nullnorm على الترتيب الشبكى.

وقد إشتملت هذه الرسالة على ستة فصول كالتالى:

الفصل الأول: يشمل هذا الفصل على مقدمة مختصرة عن موضوع الرسالة والدوافع وراء هذه البحث والهدف من ورائه مع عرض لمحتويات الرسالة.

الفصل الثانى: يقدم هذا الفصل لمحة مختصرة حول طريقة الجمع المعتمد على الترتيب الشبكى فى تكوين الترتيب الجزئى والترتيب الشبكى.

الفصل الثالث: يقدم هذا الفصل دراسة إستقصائية حول أهم وأشهر عمليات التجميع الدامجة من تعريفات وخصائص وكذلك طرق التكوين المختلفة على الترتيب الشبكى المحدود.

الفصل الرابع: يقدم هذا الفصل الطرق المقترحه لتكوين العملية nullnorm ومن خلالها تكوين العملية idempotent nullnorm وكذلك العمليتين t-norm ، t-conorm على الترتيب الشبكى.

الفصل الخامس: يقدم هذا الفصل الطرق المقترحه لتكوين العملية uninorm ومن خلالها تكوين العمليات idempotent uninorm و t-norm وكذلك t-conorm على الترتيب الشبكى المحدود وكذلك التحسينات التى يمكن عملها مستقبلا فى هذا الموضوع.

الفصل السادس: يشمل هذا الفصل النتائج التى تم الحصول عليها في هذه الرساله والأعمال المستقبلية المقترحة.

وفى نهاية الرسالة يوجد قائمة بالمراجع وعدد من الملاحق.

جامعة بنها
كلية الهندسة ببنها
قسم العلوم الهندسية الأساسية

عن الجمع المعتمد على الترتيب الشبكى ونتائجة على إنشاء عمليات تجميع دامجة

الرسالة مقدمة من

المهندس / محمود عطيه محمود إبراهيم خطاب
بكالوريوس الهندسة والتكنولوجيا فى الهندسة الكهربية

كجزء من متطلبات الحصول على درجة ماجستير العلوم الهندسية فى الرياضيات الهندسية

تحت اشراف

أ.د / على نصر الوكيل
أستاذ الرياضيات الهندسية غيرالمتفرغ
كلية الهندسة ببنها
جامعة بنها

أ.م.د / معتز صالح الزكى
أستاذ الرياضيات الهندسية المساعد
كليـــة الهندســــة
جامعة دمياط

بنها ٢٠١٩